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Cohomology of The Grassmannian

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Vector bundles are geometric objects obtained by attaching a real vector space to each point of a given topological space, called the base space, such that these spaces vary continuously. Vector bundles arise in many areas of geometry and analysis, the most notable example being perhaps the tangent bundle of a smooth manifold. In this work we will focus on the special class of complex vector bundles, which are obtained by imposing a complex structure on the real vector spaces in a given bundle.

Two central tools in the study of vector bundles are characteristic classes and a classifying space called the Grassmannian. Characteristic classes are natural associations of cohomology classes of the base space to each vector bundle. The main characteristic classes of complex vector bundles are called Chern classes, and they are even-dimensional integral cohomology classes. The Grassmannian, on the other hand, is constructed as the set of subspaces of a fixed dimension of the infinite-dimensional complex vector space $\mathbb{C}^\infty$, and it comes equipped with a tautological vector bundle.

In this work we define complex vector bundles and finite and infinite versions of the Grassmannian, and discuss the classifying space nature of the infinite Grassmannian. Then we prove the Thom isomorphism theorem concerning cohomology groups of vector bundles, and use the result to define Chern classes. Finally, we show that the integral cohomology ring of the Grassmannian is a polynomial ring generated by the Chern classes of the tautological bundle.

Keywords:	Complex vector bundle, Grassmannian, Cohomology, Chern class, Thom isomorphism
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Vektorikimput ovat geometrisia objekteja, jotka voidaan rakentaa kiinnittämällä euklidinen avaruus jonkin topologisen avaruuden, pohja-avaruuden, jokaiseen pisteeseen jatkuvalla tavalla. Vektorikimput ovat keskeisiä monilla geometrian ja analyysin alueilla, ja kenties tärkein esimerkki vektorikimpusta on sileän moniston tangenttikimppu. Tässä työssä keskitytään kompleksisiin vektorikimppuihin, jotka saadaan määrittelemällä kompleksinen rakenne annetun vektorikimpun säikeissä.

Kaksi keskeistä työkalua vektorikimppujen tutkimuksessa ovat karakteristiset luokat ja Grassmannin avaruutena tunnettu luokitteluavaruus. Karakteristinen luokka on sääntö, joka liittää jokaiseen vektorikimppuun pohja-avaruuden kohomologialuokan luonnollisella tavalla. Kompleksisten vektorikimppujen pääasiallisia karakteristisia luokkia kutsutaan Chernin luokiksi. Grassmannin avaruus puolestaan on ääretönulotteisen kompleksisen vektoriavaruuden $\mathbb{C}^\infty$ tiettyä dimensiota olevien aliavaruuksien joukko. Grassmannin avaruuteen liitetään myös niin kutsuttu tautologinen vektorikimppu.

Tässä työssä määritellään kompleksiset vektorikimput ja Grassmannin avaruuden äärellinen ja ääretön versio sekä kuvataan tapa, jolla ääretön Grassmannin avaruus voidaan ymmärtää luokitteluavaruutena. Tämän jälkeen todistetaan vektorikimppujen kohomologiaryhmiä koskeva Thomin isomorfismilause, ja käytetään kyseistä tulosta Chernin luokkien määrittelemiseen. Lopuksi näytetään, että Grassmannin avaruuden kokonaislukukertoiminen kohomologiarengas on tautologisen kimpun Chernin luokkien virittämä polynomirengas.

Asiasanat:	Kompleksinen vektorikimppu, Grassmannin avaruus, Kohomologia, Chernin luokka, Thomin isomorfismi
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Chapter 1

Introduction

Vector bundles are geometric objects constructed by attaching a vector space to each point of a given topological space. More formally, a real vector bundle is a continuous map $\pi : E \to B$ of topological spaces, such that the fiber over each point of B has the structure of a real vector space, and that over sufficiently small open neighborhoods U of B, the preimage of U in E looks like the product $U \times \mathbb{R}^n$ for some integer n. If this integer is the same for all neighborhoods U, then it is called the rank of the bundle.

Vector bundles are natural objects in many areas of geometry and analysis. Perhaps the most important example of a vector bundle is the tangent bundle TM of a smooth manifold M, which is constructed by gluing to a point $p \in M$ the tangent space T_pM in such a way that the tangent spaces vary smoothly over the manifold. The tangent bundle is the natural environment to endow M with additional geometric structure. For example, a Riemannian metric on M is a smoothly varying choice of inner product at each tangent space T_pM. Another central example is the cotangent bundle T^*M and its exterior products, which form the basis of the de Rham complex and de Rham cohomology. A special class of vector bundles are the complex vector bundles, which locally look like products $U \times \mathbb{C}^n$. These arise naturally for example in the study of complex analytic spaces and complex varieties. In this work we will mainly be interested in complex vector bundles.

There is a natural notion of a vector bundle isomorphism, preserving both the topological and the linear structure. One then faces the following classification question of vector bundles. Given a space B, describe all isomorphism classes of vector bundles over B of some fixed rank n. This question leads to the construction of classifying spaces of vector bundles, called Grassmannians. The classifying space of complex vector bundles of rank n is the complex Grassmannian G_n, and it comes equipped with a canonical complex vector bundle over it, called the tautological bundle. Using G_n we can now give a classification of complex vector bundles as follows. If $\pi : E \to B$ is a vector bundle over a paracompact space B, there exists a continuous map $f : B \to G_n$ such that E is the pullback of the tautological bundle under f. Furthermore, two bundles over B are isomorphic if and only if the corresponding maps $B \to G_n$ are homotopic. In other words, isomorphism classes of rank n complex vector bundles over B are in one-to-one correspondence with the homotopy classes of maps $B \to G_n$.

The complex Grassmannian is a generalization of the familiar complex projective space. As a set, the Grassmannian G_n is the collection of n-dimensional subspaces of $\mathbb{C}^\infty$, the direct sum of a countably infinite number of copies of the complex numbers. It can be given a natural topology using an auxiliary space called the Stiefel space V_n, which consists of orthonormal n-tuples of vectors in $\mathbb{C}^\infty$. There is a canonical map $V_n \to G_n$, sending an n-tuple to the hyperplane it spans, and we endow G_n with the quotient topology defined by this map. Having introduced a topology, we can now for example speak about continuous families of vector spaces parametrized by G_n.

Cohomology provides a tool to differentiate between isomorphism classes of vector bundles over a given base space. The main cohomology invariants of vector bundles are called characteristic classes.

They are natural associations of cohomology classes of the base space B to each vector bundle over B. An implication of the classifying space nature of the Grassmannian is that characteristic classes are in one to one correspondence with cohomology classes of the Grassmannian. Thus, the calculation of the cohomology ring of the Grassmannian becomes a central task in studying vector bundles. The main characteristic classes of complex vector bundles are called Chern classes, and the aim of this work is to define these classes and show that the integral cohomology ring of the complex Grassmannian is a polynomial ring generated by the Chern classes associated to the tautological bundle.

There are also finite versions of the complex Grassmannian. If k is an integer, $k \geq n$, we define the Grassmannian $G_n(\mathbb{C}^k)$ as the set of n-dimensional subspaces of $\mathbb{C}^k$. It can be given a topology in the same way as the infinite Grassmannian. However, the Grassmannians have more natural geometric structure than mere topology. In this work, we will show that the finite complex Grassmannian $G_n(\mathbb{C}^k)$ is a topological manifold of dimension $2n(k-n)$, but in fact it has the structure of a complex analytic space in a natural way. Furthermore, we will describe CW structures in both the finite and the infinite case. The CW decomposition is formed by the so-called Schubert cells, defined by considering how the n-dimensional subspaces of $\mathbb{C}^k$ intersect with a given sequence of subspaces. The decomposition into Schubert cells gives rise to an intersection theory in homology called Schubert calculus. For the complex analytic structure and Schubert calculus, see section 1.5 of [4]. For an application of Schubert calculus to eigenvalue problems of Hermitian matrices, see [8]. As another example, [2] gives an application of Schubert calculus to interference alignment problems in certain wireless communication systems.

Cohomology of the finite Grassmannian $G_n(\mathbb{C}^k)$ can also be accessed using Hodge theory. In Hodge theory, one studies the connection of de Rham cohomology of a Riemannian manifold and harmonic differential forms associated to a Laplacian operator arising from the Riemannian metric. In the case of the complex Grassmannian, there is a unique Kähler metric satisfying an invariance condition under the action of a unitary group. It then turns out that the Chern classes of a $GL_n(\mathbb{C})$-principal bundle over $G_n(\mathbb{C}^k)$ are represented by certain harmonic forms, that these representatives are algebraically independent, and that any harmonic form can be represented algebraically by the Chern classes. See chapter V of [3] for details.

The Grassmannians play an important role in algebraic geometry. Firstly, there is a classical embedding of the finite Grassmannian into complex projective space such that the image is a complete smooth variety. This is called the Plücker embedding, and it can be described as follows. An n-dimensional subspace of $\mathbb{C}^k$ is determined by n linearly independent vectors $v_1, ..., v_n \in \mathbb{C}^k$. The Plücker embedding

$$p : G_n(\mathbb{C}^k) \to \mathbb{P}(\bigwedge^n \mathbb{C}^k) = \mathbb{CP}^{\binom{k}{n}-1}$$

maps the Grassmannian to the nth exterior product of $\mathbb{C}^k$ by sending the plane spanned by $v_1, ..., v_n$ to the wedge product $v_1 \wedge \cdots \wedge v_n$. It can be shown that the image is the zero set of a collection of quadratic equations, so the Grassmannian embeds as the intersection of quadrics. For example, the Grassmannian $G_2(\mathbb{C}^4)$ can be realized as the variety in $\mathbb{CP}^5$ whose equation is

$$x_0x_1 - x_2x_3 + x_4x_5 = 0.$$

For more details, see again [4].

Grassmannians are important examples of moduli spaces. In informal terms, a moduli space is a space that parametrizes a given class of geometric objects. More precisely, if $\mathcal{C}$ is a class of geometric objects (such as algebraic curves, varieties, or vector bundles over a given space), then a fine moduli space for $\mathcal{C}$ is a space $\mathcal{M}$ whose points correspond to objects in $\mathcal{C}$, or more precisely, there is a family $U \to \mathcal{M}$ whose fibers are the objects of $\mathcal{C}$. Furthermore, this family is universal in the sense that if $U' \to B$ is a family of objects in $\mathcal{C}$ over B, then there exists a map $B \to \mathcal{M}$ such that U' can be recovered as the pullback of U by this map. The Grassmannian $G_n(\mathbb{C}^k)$ is the moduli space n-dimensional subspaces of the complex vector space $\mathbb{C}^k$, and the universal family is the tautological bundle. More generally, Grassmannians can be defined over any ring, or even over any scheme, parametrizing locally free sheaves. For more details, see sections 6.7 and 16.7 of [17]. For introduction to moduli spaces of curves with a brief discussion on Grassmannians, see [16].

Apart from those mentioned above, Grassmannians and their generalizations have applications in various other fields of natural sciences. For example, [1] describes a generalization of the Grassmannian, called the amplituhedron, for calculating scattering amplitudes in particle physics. [15] discusses statistical methods on Grassmannian and Stiefel manifolds applied to computer vision.

This work is organized as follows. In chapter 2, we make some brief remarks on various topological notions that will appear later, and then move on to a more detailed discussion of singular homology and cohomology theories. In chapter 3, we define the main geometric objects of this work, the complex Grassmannians, both in the finite and the infinite case. We prove some of their most basic topological properties, and then describe the CW decomposition into Schubert cells. In chapter 4, we introduce real vector bundles and discuss their properties and operations between vector bundles. Then we define complex vector bundles, construct the tautological bundles over the Grassmannians, and explain how the infinite Grassmannian can be seen as the classifying space of complex vector bundles. In chapter 5, we combine vector bundles and singular cohomology with the aim of describing the cohomology ring of the infinite Grassmannian. To achieve this, we first state and prove the Thom isomorphism theorem and use it to define the Euler class and Chern classes.

As our main source we have used the classic book *Characteristic Classes* by J. Milnor and J. Stasheff [13]. For algebro-topological background, we have consulted *Algebraic Topology* by A. Hatcher [6]. Other general references in this subject are for example Fibre Bundles by D. Husemoller [9], and Vector Bundles and K-theory by A. Hatcher [7].

Chapter 2

Preliminaries

In this preliminary section we first present some concepts from general topology and state some results that will appear in the course of discussion of vector bundles and Grassmannians. We will omit most proofs. After that, we will discuss in some length and detail the basic notions of singular homology and cohomology, beginning with rudiments of homological algebra. For a general reference on topology, see [14]. For homology and cohomology, see [6].

2.1 Some Topological Notions

Before going into more sophisticated notions, we will state an extremely elementary property of continuous functions which will however appear several times in what follows. Namely, if $f : X \to Y$ is a map between topological spaces, and if $\{U_\alpha\}$ is an open cover of X, then f is continuous if and only if the restriction $f|_{U_\alpha} : U_\alpha \to Y$ is continuous for all U_α.

2.1.1 Homotopy

Homotopy is a concept that makes the idea of continuously deforming spaces or maps between spaces precise. Two continuous maps $f_0, f_1 : X \to Y$ are called **homotopic**, denoted $f_0 \simeq f_1$, if there exists a continuous map $F : X \times I \to Y$, where $I = [0, 1]$, such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$ for all $x \in X$. Two spaces X and Y are called **homotopy equivalent** if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_X$ and $g \circ f \simeq \mathrm{id}_Y$. The maps f and g are called **homotopy equivalences**.

One important special case of homotopy is deformation retract. Let X be a topological space and let $A \subset X$ be a subspace. A is a **deformation retract** of X, if there exists a continuous map $F : X \times I \to X$ such that $F(x, 0) = x$ for all $x \in X$, $F(a, t) = a$ for all $a \in A$ and $t \in [0, 1]$, and $F(x, 1) \in A$ for all $x \in X$.

Homotopy equivalence is an equivalence relation, so it gives a partition of topological spaces into equivalence classes called **homotopy types**. As an example, spaces with the homotopy type of a point are called **contractible**.

2.1.2 Direct Limit Topology

Given a sequence of topological spaces $X_1 \subset X_2 \subset X_3 \subset \ldots$, the union $X = \cup_{n=1}^{\infty} X_n$ is said to have the **direct limit topology** or the **weak topology**, if a set $U \subset X$ is open if and only if $U \cap X_n$ is open in X_n for all n. With this topology, a map $f : X \to Y$ is continuous if and only if the restriction $f|_{X_n} : X_n \to Y$ is continuous for all n.

A topological space X is a called **locally compact**, if for every point $p \in X$ there exists a compact set K containing some open neighborhood of p. We have the following result. For a proof, see p. 64 of [13].

Proposition 2.1.1. *Let $A_1 \subset A_2 \subset ...$ and $B_1 \subset B_2 \subset ...$ be two sequences of locally compact spaces with direct limits A and B respectively. The product topology on $A \times B$ is the same as the direct limit topology arising from the sequence $A_1 \times B_1 \subset A_2 \times B_2 \subset ...$.*

2.1.3 Manifolds and CW Complexes

We now describe two particularly important classes of spaces, namely topological manifolds and CW complexes.

A topological space X is called a **Hausdorff** space if for any two distinct points x and y there exist open neighborhoods U_x and U_y, respectively containing x and y, such that $U_x \cap U_y = \emptyset$. Note that X is Hausdorff if for any distinct points x and y there exists a continuous function $f : X \to \mathbb{R}$ such that $f(x) \neq f(y)$, for then distinct open neighborhoods of $f(x)$ and $f(y)$ have distinct open preimages in X. A space X is **second countable** if the topology of X has a countable basis, meaning that there is a countable collection $\mathcal{B}$ of open sets such that every open set of X is a union of sets in $\mathcal{B}$. A **topological manifold** is a second countable Hausdorff space where every point has an open neighborhood homeomorphic to an open set of a Euclidean space. If M is a manifold and every point of M has a neighborhood homeomorphic to an open set of $\mathbb{R}^n$, then the **dimension** of M is n. See [10] for more on topological manifolds.

A CW complex is a space constructed by gluing cells of different dimensions together in such a way that the attaching information reflects the geometric structure of the resulting space. We will give the definition of CW complexes in terms of cell decompositions, and then describe an inductive process to construct CW complexes.

A **closed cell** of dimension n is any space homeomorphic to the closed ball $\overline{B}^n$, and an **open cell** of dimension n is a any space homeomorphic to the open ball B^n, that is, the interior of $\overline{B}^n$. A **CW complex** is a Hausdorff space X together with a collection of maps $\Phi_\alpha : D^n_\alpha \to X$, where D^n_α is a closed cell of dimension $n = n(\alpha)$ depending on the index α. These maps must satisfy the following conditions.

(i) Each Φ_α restricts to a homeomorphism from $\operatorname{int} D^n_\alpha$ onto a set $e^n_\alpha \subset X$, called a **cell**. These cells are disjoint and cover X.
(ii) For each α, the image of the boundary of D^n_α is contained in the union of a finite number of cells of dimension less than n.
(iii) A subset of X is closed if and only if it meets the closure of each cell of X in a closed set.

The map Φ_α is called the **characteristic map** of the cell e^n_α. The union of cells of dimension at most n is called the n**-skeleton** of X and is denoted by X^n. Thus, the skeleta of X form a nested sequence $X^0 \subset X^1 \subset X^2 \subset ...$, and X is the union of all its skeleta. If X has only finitely many cells, the maximal dimension of its cells is called the **dimension** of X. In this case, the third condition is automatically satisfied. If X is any CW complex, then a finite union of cells of X that is itself a CW complex with the same characteristic maps is called a **finite subcomplex**. A central property of the topology on a CW complex is that every compact subspace is contained in a finite subcomplex. For more on CW complexes, see [6].

2.1.4 Paracompact Hausdorff Spaces

A stronger separation property than being Hausdorff is normality. A Hausdorff space X is **normal** if for any disjoint closed subsets $V, V' \subset X$ there exist open sets $U, U' \subset X$ such that $V \subset U$, $V' \subset U'$, and $U \cap U' = \emptyset$. The next result is of fundamental importance in topology.

Theorem 2.1.2 (Urysohn's Lemma)**.** *Let X be a normal space and let $A, B \subset X$ be disjoint closed set. There exists a continuous function $f : X \to [0,1]$ such that $f|_A \equiv 0$ and $f|_B \equiv 1$.*

Urysohn's Lemma implies the existence of bump functions in normal spaces.

Corollary 2.1.3. *Let X be a normal space. If $A \subset X$ is a closed set and $U \subset X$ is an open set containing A, then there exists a continuous function $X \to [0,1]$ such that $f|_A \equiv 1$ and $f|_{X \setminus U} \equiv 0$.*

Recall that a topological space X is compact if every open cover of X has a finite subcover. We will next describe an important generalization of compactness. Let X be a topological space. A **refinement** of an open cover $\{U_\alpha\}$ is another open cover $\{V_\beta\}$ such that for each V_β there exists some U_α such that $V_\beta \subset U_\alpha$. A collection $\mathcal{A}$ of subset of X is **locally finite** if each point of X has a neighborhood that intersects only a finite number of sets in $\mathcal{A}$. We say that X is **paracompact** if every open cover of X has a locally finite refinement. By combining paracompactness with the Hausdorff property, we obtain the following results.

Proposition 2.1.4. *Every paracompact Hausdorff space is normal.*

Proposition 2.1.5. *Let X be paracompact and Hausdorff and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of X. There exists a locally finite refinement $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ of $\mathcal{U}$, indexed by the same set $\mathcal{A}$, such that $\overline{V}_\alpha \subset U_\alpha$ for all α.*

Many familiar topological spaces, for example all manifolds and all CW complexes, are paracompact Hausdorff spaces. For proofs and further properties, see p. 109-114 of [10].

2.1.5 Path-Connectedness of the Complex General Linear Group

We conclude these topological remarks with the following result that will be used in a few instances later on. Recall that the complex general linear group $GL_n(\mathbb{C})$ is the set of invertible $n \times n$ complex matrices. By considering each matrix as a complex vector of length n^2, we give $GL_n(\mathbb{C})$ the subspace topology inherited from $\mathbb{C}^{n^2}$.

Theorem 2.1.6. *The complex general linear group $GL_n(\mathbb{C})$ is path-connected.*

Proof. Let $A \in GL_n(\mathbb{C})$. By the Schur decomposition, A is similar to an upper triangular matrix, so we have $A = C^{-1}BC$ for some invertible upper triangular matrix B. Define $B(t)$ by multiplying every entry of B above the diagonal by $1-t$. When $0 \leq t \leq 1$, the matrices $B(t)$ form a continuous path of invertible matrices, since $\det(B(t)) = \det(B) = \det(A)$ for all t. $B(1)$ is a diagonal matrix with nonzero diagonal entries λ_i, so we can find paths $[1,2] \to \mathbb{C}$ from λ_i to 1 of nonzero complex numbers. These paths together define a path from $B(1)$ to I through invertible matrices. Conjugating by C and traversing these two paths consecutively yields a path from A to I. $\square$

See [5] for further information on matrix Lie groups.

2.2 Homology and Cohomology

Our goal is to study vector bundles using certain natural associations of cohomology classes called characteristic classes. In this chapter we will describe the required algebro-topological background by defining singular homology and cohomology theories and stating some of their properties. We begin with some homological algebra. All definitions and proofs can be found in [6].

2.2.1 Elements of Homological Algebra

A **chain complex** of abelian groups, denoted F_*, is a sequence

$$\cdots \xrightarrow{f_{n+2}} F_{n+1} \xrightarrow{f_{n+1}} F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots$$

of abelian groups F_n and homomorphisms f_n, such that the latter satisfy the relation $f_n \circ f_{n+1} = 0$ for all n. This is equivalent with having $\operatorname{im} f_{n+1} \subset \ker f_n$. The maps f_n are called the **boundary maps** of the complex, collectively denoted by f_*. Since both im f_{n+1} and ker f_n are subgroups of the abelian group F_n, we can form the quotient group

$$H_n(F_*) = \ker f_n / \operatorname{im} f_{n+1},$$

called the n**th homology group** of the chain complex. A **chain map** between chain complexes F_* and G_* is a sequence of homomorphism $\phi_n : F_n \to G_n$ that commute with the boundary maps, that is $g_n \circ \phi_n = \phi_{n-1} \circ f_n$. More generally, a **chain map of degree** d is a sequence of maps $\phi_n : F_n \to G_{n+d}$ that commute with the boundary maps.

An **exact sequence** is a chain complex satisfying $\operatorname{im} f_{n+1} = \ker f_n$, or equivalently $H_n(F_*) = 0$, for all n. For example, exactness of $0 \to A \xrightarrow{f} B$ implies that f is an injection, and similarly exactness of $A \xrightarrow{f} B \to 0$ implies that f is a surjection. An exact sequence of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is called a **short exact sequence**. In this case, exactness implies that $A \to B$ is injective, $B \to C$ is surjective, and C is isomorphic to B/A when we identify A with its image in B.

A **short exact sequence of chain complexes** is a pair of chain maps $0 \to A_* \xrightarrow{i} B_* \xrightarrow{j} C_* \to 0$ such that each of the sequences $0 \to A_n \xrightarrow{i_n} B_n \xrightarrow{j_n} C_n \to 0$ is exact. A short exact sequence of chain complexes gives rise to a **long exact sequence of homology groups**

$$\cdots \xrightarrow{\partial_{n+1}} H_n(A_*) \xrightarrow{i_*} H_n(B_*) \xrightarrow{j_*} H_n(C_*) \xrightarrow{\partial_n} H_{n-1}(A_*) \xrightarrow{i_*} \cdots.$$

Here the homomorphisms i_* and j_* are induced by the maps i and j, and the **connecting homomorphisms** $\partial_n : H_n(C_*) \to H_{n-1}(A_*)$ are defined as follows. Let f_*, g_* and h_* be the boundary maps of A_*, B_* and C_*, respectively. Let $\tilde{x} \in H_n(C_*)$ be represented by $x \in \ker h_n \subset C_n$. Since j_n is surjective, there exists $y \in B_n$ such that $j_n(y) = x$. Then $g_n(y)$ is in $\ker j_{n-1} = \operatorname{im} i_{n-1}$ since

$$j_{n-1} g_n(y) = h_n j_n(y) = h_n(x) = 0,$$

so there exists $z \in A_{n-1}$ such that $i_{n-1}(z) = g_n(y)$. It can be easily shown that z is in $\ker f_n$. Now define $\partial_n(\tilde{x}) = \tilde{z}$, where $\tilde{z} \in H_n(A_*)$ is the homology class of z. We will not prove that the connecting homomorphism is well-defined or that the resulting sequence is exact.

The long exact sequence is **natural** in the sense that if we have another short exact sequence

$$0 \to A'_* \xrightarrow{i'} B'_* \xrightarrow{j'} C'_* \to 0$$

together with homomorphisms $A_n \xrightarrow{a_n} A'_n, B_n \xrightarrow{b_n} B'_n$ and $C_n \xrightarrow{c_n} C'_n$ which commute with boundary maps and the maps i, j, i' and j', then there are induced maps a_*, b_* and c_* such that the diagram

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & H_n(A_*) \arrow[r, "i_*"] \arrow[d, "a_*"] & H_n(B_*) \arrow[r, "j_*"] \arrow[d, "b_*"] & H_n(C_*) \arrow[r, "\partial"] \arrow[d, "c_*"] & H_{n-1}(A_*) \arrow[r, "i_*"] \arrow[d, "a_*"] & \cdots \\
\cdots \arrow[r, "\partial'"] & H_n(A_*) \arrow[r, "i'_*"] & H_n(B_*) \arrow[r, "j'_*"] & H_n(C_*) \arrow[r, "\partial'"] & H_{n-1}(A_*) \arrow[r, "i'_*"] & \cdots
\end{tikzcd}$$

commutes.

Given an abelian group G, we can form the dual complex of a chain complex by defining

$$F_n^* = \operatorname{Hom}(F_n, G)$$

and defining the **coboundary maps** $f_n^* : F_{n-1}^* \to F_n^*$ by precomposing a given $\phi \in F_{n-1}^*$ with f_n. The resulting sequence

$$\cdots \xleftarrow{f_{n+2}^*} F_{n+1}^* \xleftarrow{f_{n+1}^*} F_n^* \xleftarrow{f_n^*} F_{n-1}^* \xleftarrow{f_{n-1}^*} \cdots$$

is a chain complex. The homology groups of this complex, denoted by $H^n(F_*; G)$, are called the **cohomology groups with coefficients in** G of the original complex. If the groups in F_* are free, the relationship between homology and cohomology groups is given by the following.

Theorem 2.2.1 (Universal Coefficient Theorem of Cohomology)**.** *The following sequence is exact.*

$$0 \to \mathrm{Ext}(H_{n-1}(F_*), G) \to H^n(F_*; G) \to \mathrm{Hom}(H_n(F_*), G) \to 0$$

We also define the **homology groups with coefficients in** G, denoted $H_n(F_*; G)$, as the homology groups associated to the chain complex associated by tensoring each F_n with G. Similarly with Theorem 2.2.1, we have the following.

Theorem 2.2.2 (Universal Coefficient Theorem of Homology)**.** *The following sequence is exact.*

$$0 \to H_n(F_*) \otimes G \to H_n(F_*; G) \to \mathrm{Tor}(H_{n-1}(F_*), G) \to 0$$

We will only briefly comment on the Ext and Tor functors without giving a precise definition of them or the maps appearing in the universal coefficient theorems. Category theoretically Ext is the first right derived functor of the Hom functor, and, dually, Tor is the first left derived functor of the tensor product functor. We will only need the following property enjoyed by both Ext and Tor: if either F or G is a free abelian group, then $\mathrm{Ext}(F, G) = 0$ and similarly $\mathrm{Tor}(F, G) = 0$. It now follows that if $H_{n-1}(F_*)$ is a free abelian group, then the two universal coefficient theorems reduce to isomorphism

$$H^n(F_*; G) \cong \mathrm{Hom}(H_n(F_*), G) \qquad \text{and} \qquad H_n(F_*) \otimes G \cong H_n(F_*; G).$$

In addition, we remark that similar universal coefficient theorems hold if we replace abelian groups with modules over a commutative ring R.

As an illustration of homological algebra, we will now prove a result that will be important in the proof of Theorem 5.1.2. We will use the following definition. Given a chain map $\phi : (A_*, \partial) \to (B_*, \delta)$ of degree d, the **mapping cone** of ϕ is the chain complex $(C(\phi)_*, \partial^\phi)$, where $C(\phi)_n = A_{n-d-1} \oplus B_n$ and the boundary map is defined by $\partial_n^\phi(a, b) = (-\partial a, \phi(a) + \delta b)$. The mapping cone is indeed a chain complex, since

$$\begin{aligned}(\partial^\phi)^2(a, b) = \partial^\phi(-\partial a, \phi(a) + \delta b) &= (\partial^2 a, -\phi(\partial a) + \delta\phi(a) + \delta^2 b) \\ &= (\partial^2 a, -\phi(\partial a) + \phi(\partial a) + \delta^2 b) = (0, 0).\end{aligned}$$

The complex $C(\phi)_*$ fits in the short exact sequence of chain complexes

$$0 \to B_* \to C(\phi)_* \to A_* \to 0,$$

where the first map is the inclusion $b \mapsto (0, b)$ and the second map is the projection $(a, b) \mapsto a$. The induced long exact sequence of homology groups is then

$$\cdots \to H_{n+1}(C(\phi)_*) \to H_{n-d}(A_*) \to H_n(B_*) \to H_n(C(\phi)_*) \to \cdots,$$

where the connecting homomorphism $H_{n-d}(A_*) \to H_n(B_*)$ is given by ϕ_*. We now deduce that ϕ_* is an isomorphism for all n if and only if $H_*(C(\phi)_*) = 0$.

Proposition 2.2.3. *Let A_* and B_* be chain complexes of free abelian groups. If a chain map $\phi : A_* \to B_*$ induces isomorphisms of cohomology groups $H^n(A_*;\Lambda) \to H^n(B_*;\Lambda)$ for all n and all coefficient fields Λ, then it induces isomorphisms of homology and cohomology groups with arbitrary coefficients.*

Proof. Using the mapping cone $F_* = C(\phi)_*$, we must prove that if $H^n(F_*;\Lambda) = 0$ for all fields Λ, then $H_n(F_*;G) = H^n(F_*;G) = 0$ for all abelian groups G. Denote the boundary map of F_* by ∂. For a field Λ,

$$\mathrm{Ext}(H_{n-1}(F_*),\Lambda) = \mathrm{Tor}(H_{n-1}(F_*),\Lambda) = 0,$$

and it follows from the universal coefficient theorem that

$$H_n(F_*;\Lambda) \cong H_n(F_*) \otimes \Lambda \quad \text{and} \quad H^n(F_*;\Lambda) \cong \mathrm{Hom}(H_n(F_*),\Lambda).$$

Using adjointness of Hom and $\otimes$, we have

$$\begin{aligned} \mathrm{Hom}_\Lambda(H_n(F_*;\Lambda),\Lambda) &\cong \mathrm{Hom}_\Lambda(H_n(F_*)\otimes\Lambda,\Lambda) \cong \mathrm{Hom}(H_n(F_*),\mathrm{Hom}(\Lambda,\Lambda)) \\ &\cong \mathrm{Hom}(H_n(F_*),\Lambda) \cong H^n(F_*;\Lambda) = 0 \end{aligned}$$

Since $H_n(F_*;\Lambda)$ is a vector space over Λ, we must have $H_n(F_*;\Lambda) = 0$ since otherwise there would exist a nontrivial homomorphism $H_n(F_*;\Lambda) \to \Lambda$.

In particular, we have $H_n(F_*;\mathbb{Q}) = 0$ and $H_n(F_*,\mathbb{F}_p) = 0$ for all primes p, where $\mathbb{F}_p$ denotes the field of p elements. We will first prove that $H_n(F_*) = 0$. Let $\sigma \in \ker\partial$. Then $\sigma\otimes 1 \in \ker\partial\otimes\mathbb{Q} = \mathrm{im}\,\partial\otimes\mathbb{Q}$, so for some $\sigma_i \in \mathrm{im}\,\partial$ and $k_i/m \in \mathbb{Q}$,

$$\sigma\otimes 1 = \sum \sigma_i \otimes (k_i/m) \quad \Rightarrow \quad m\sigma\otimes 1 = \sum (k_i\sigma_i)\otimes 1,$$

which shows that $m\sigma \in \mathrm{im}\,\partial$. Hence every element in $H_n(F_*)$ is a torsion element. To show that $H_n(F_*) = 0$, we must show that each element of prime order p is zero. If $\sigma \in \ker\partial$ represents such an element, then $p\sigma = \partial\tau$ for some $\tau \in F_{n+1}$. In $F_*\otimes\mathbb{F}_p$ we then have $\partial\tau\otimes 1 = p\sigma\otimes 1 = 0$, and since $\ker\partial\otimes F_p = \mathrm{im}\,\partial\otimes\mathbb{F}_p$, it follows that $\partial\tau \in \mathrm{im}\,\partial\otimes\mathbb{F}_p$. Hence, for some $k_i \in \mathbb{Z}, \tau_i \in F_{n+1}, \nu_i \in F_n$ and $s_i \in \mathbb{F}_p$, we can write

$$\tau\otimes 1 = \sum k_i(\partial\tau_i + p\nu_i)\otimes s_i = (\sum s_ik_i\partial\tau_i + s_ip\nu_i)\otimes 1.$$

Thus, $\tau = \partial\rho + p\nu$, where $\rho = \sum s_ik_i\tau_i$ and $\nu = \sum s_i\nu_i$. Now, $p\sigma = \partial\tau = \partial^2\rho + p\partial\nu = p\partial\nu$, and hence $\sigma = \partial\nu$. This proves that $H_n(\mathbb{F}_*) = 0$. The result follows now immediately from the two universal coefficient theorems. □

The following is an important result of homological algebra that we will use a few times. The proof is an elementary but rather lengthy exercise of a method called diagram chasing.

Lemma 2.2.4 (Five lemma)**.** *Assume that in the commutative diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[r] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

the rows are exact, and the maps α, β, δ and ϵ are isomorphisms. Then also γ is an isomorphism.

2.2.2 Limits and colimits

We will now discuss briefly the concepts of limit and colimit of groups.

A **directed set** is a partially ordered set I such for each $i, j \in I$ there exists some $k \in I$ such that $i, j \leq k$. A **directed system of groups** is a collection of groups $\{G_i\}_{i \in I}$ indexed by a directed set I such that for each $i, j \in I$ with $i \leq j$, there exists a homomorphism $f_{ij} : G_i \to G_j$. In addition, these homomorphisms must satisfy the conditions that $f_{ii} = \mathrm{id}_{G_i}$ and $f_{jk} \circ f_{ij} = f_{ik}$. The **direct limit** of such a system, denoted by $\varinjlim G_i$, is defined as follows. As a set it is the quotient of the disjoint union $\coprod_{i \in I} G_i$ such that $a \in G_i$ and $b \in G_j$ are equivalent if and only if $f_{ik}(a) = f_{jk}(b)$ for some $k \in I$. Since any two classes $[a]$ and $[b]$ have representatives a', b' in some G_k, we have a well-defined group operation given by $[a] + [b] = [a' + b']$. For each $i \in I$, there is a natural map $G_i \to \varinjlim G_i$ sending $a \in G_i$ to $[a] \in \varinjlim G_i$.

The inverse limit is dual to the direct limit. Given a directed set I, an **inverse system of groups** is a collection of groups $\{G_i\}_{i \in I}$ indexed by a directed set I such that for each $i, j \in I$ with $i \leq j$, there exists a homomorphism $f_{ij} : G_j \to G_i$. These homomorphisms must again satisfy the conditions that $f_{ii} = \mathrm{id}_{G_i}$ and $f_{ij} \circ f_{jk} = f_{ik}$. The **inverse limit** $\varprojlim G_i$ of the system is the subgroup of the direct product $\prod_{i \in I} G_i$ consisting of sequences $(a_i)_{i \in I}$ such that $a_i = f_{ij}(a_j)$ for all i, j with $i \leq j$. For each $i \in I$ there is a natural map $\varprojlim G_i \to G_i$ defined as the restriction of the projection map $\prod_{i \in I} G_i \to G_i$.

A basic relation between the direct and the inverse limit is given by the following result.

Lemma 2.2.5. *Given a directed system of groups* $\{G_i\}_{i \in I}$ *and any group* H*, then*

$$\varprojlim \mathrm{Hom}(G_i, H) = \mathrm{Hom}(\varinjlim G_i, H).$$

The proof is straightforward. Namely, a homomorphism from the direct limit $\varinjlim G_i$ to H is a collection of homomorphisms $\phi_i : G_i \to H$ such that $\phi_j = \phi_i \circ f_{ji}$ for all $j \geq i$, which is exactly the data of an element of $\varprojlim \mathrm{Hom}(G_i, H)$.

Direct and inverse limits satisfy the following universal properties. Let $\{G_i\}_{i \in I}$ be a directed system of groups together with maps $f_{ij} : G_i \to G_j$, and let $h_i : G_i \to \varinjlim G_i$ be the natural maps. If there exist maps $g_i : G_i \to H$ to some group H satisfying $g_i = g_j \circ f_{ij}$ whenever $i \leq j$, then these maps factor uniquely through $\varinjlim G_i$. In other words, there exists a unique map $g : \varinjlim G_i \to H$ such that $g_i = g \circ h_i$. Similarly, if $\{G_i\}_{i \in I}$ is an inverse system of groups, then collections of maps $g_i : H \to G_i$ satisfying analogous compatibility conditions factor uniquely through a map $H \to \varprojlim G_i$. In fact, these universal properties can be used as the definitions of direct and inverse limits.

2.2.3 Singular Homology

An n**-simplex** is the convex hull of $n + 1$ points $v_0, ..., v_n$ in $\mathbb{R}^m$ such that the vectors $v_1 - v_0, ..., v_n - v_0$ are linearly independent. An n-simplex is denoted $[v_0, ..., v_n]$, and the points v_i are called its **vertices**. To endow each n-simplex with an orientation, we consider the ordering of the vertices as part of the definition. The $(n-1)$**-faces** of $[v_0, ..., v_n]$ are the $(n-1)$-simplices $[v_0, ..., \hat{v}_i, ..., v_n]$, where $\hat{v}_j$ means omission of the jth vertex. Similarly we can define m-faces for all $0 \leq m \leq n$ by omitting all but $m + 1$ vertices. The **standard** n**-simplex** in $\mathbb{R}^{n+1}$ is the set

$$\Delta^n = \{ (t_0, ..., t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0\ \forall i \},$$

whose vertices are the standard unit vectors of $\mathbb{R}^{n+1}$. A 0-simplex is simply a point, a 1-simplex is a line segment, a 2-simplex a triangle, and a 3-simplex a tetrahedron.

Let X be a topological space. For $n \geq 0$, the n**th chain group** $C_n(X)$ is defined as the free abelian group generated by all continuous maps $\sigma : \Delta^n \to X$. Elements of $C_n(X)$ are called **singular** n**-chains** in X. The **boundary homomorphism** $\partial : C_n(X) \to C_{n-1}(X)$ is defined by linearly extending the formula

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i \sigma|[v_0, ..., \hat{v}_i, ..., v_n],$$

where $\sigma|A$ means restricting σ to A. It is straightforward to check that $\partial^2 = 0$, so we obtain a chain complex

$$\cdots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{0} 0.$$

Define the n**th singular homology group** $H_n(X)$ of X to be the nth homology group of this complex:

$$H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

Elements of $\ker \partial$ are called **cycles** and elements of $\operatorname{im} \partial$ **boundaries**.

Given an abelian group G, the **singular homology groups with coefficients in** G, denoted by $H_n(X; G)$, are defined by tensoring the singular chain groups with G and taking the homology groups of the resulting chain complex.

2.2.4 Singular Cohomology

Let now G be an abelian group. Define the n**th singular cochain group** with coefficients in G as the dual group of the nth singular chain group:

$$C^n(X; G) = \operatorname{Hom}(C_n(X), G).$$

By dualizing the boundary map $\partial : C_{n+1}(X) \to C_n(X)$, we obtain the **coboundary map**

$$\delta : C^n(X; G) \to C^{n+1}(X; G).$$

It follows that $\delta^2 = 0$, so we have a chain complex

$$0 \to C^0(X; G) \xrightarrow{\delta} C^1(X; G) \xrightarrow{\delta} \cdots \xrightarrow{\delta} C^n(X; G) \xrightarrow{\delta} C^{n+1}(X; G) \xrightarrow{\delta} \cdots.$$

The n**th singular cohomology group** $H^n(X; G)$ is defined as the nth homology group of this chain. Elements of $\ker \delta$ are called **cocycles**, and elements of $\operatorname{im} \delta$ are called **coboundaries**. The relationship between singular homology and cohomology groups is described by Theorem 2.2.1. In particular, if G is a field, or $H_{n-1}(X)$ is a free abelian group, then we have an isomorphism

$$H^n(X; G) \cong \operatorname{Hom}(H_n(X), G).$$

Note that a group homomorphism $G \to G'$ induces a homomorphism $H^n(X; G) \to H^n(X; G')$ in the obvious way.

2.2.5 Relative Homology and Cohomology Groups

Let A be a subspace of a topological space X. Define the **relative chain group** $C_n(X, A)$ to be the quotient group $C_n(X)/C_n(A)$. We can regard the relative chain group as the free abelian group generated by all continuous maps $\Delta^n \to X$ whose image is not contained in A. Since the boundary of a cycle contained in A is in A, the boundary maps descend to the quotients, so we obtain a chain complex of relative chain groups. The homology groups of this complex are called the **relative homology groups**, and are denoted

$H_n(X, A)$. The quotient map $\partial_n : C_n(X, A) \to C_{n-1}(X, A)$ is called the **relative boundary map**, and the elements of $\ker \partial$ and $\operatorname{im} \partial$ are called **relative cycles** and **relative boundaries**, respectively. By contrast to relative homology groups, the groups $H_n(X)$ are sometimes called **absolute homology groups**.

We have a short exact sequence of chain groups

$$0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X, A) \to 0,$$

where i and j are the obvious inclusion and quotient maps. This extends to a short exact sequence of chain complexes $0 \to C_*(A) \to C_*(X) \to C_*(X, A) \to 0$, so we obtain a long exact sequence

$$\cdots \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} \cdots .$$

The connecting homomorphisms $H_n(X, A) \xrightarrow{\partial_n} H_{n-1}(A)$ have the obvious geometric interpretation: a relative cycle in $C_n(X, A)$ has its boundary contained in A, so ∂_n simply takes the relative cycle to its boundary. Under certain technical assumptions, there is a close relationship between the relative homology groups $H_n(X, A)$ and the absolute homology groups $H_n(X/A)$ of the quotient space X/A.

Given an abelian group G, we define the **relative cochain group** $C^n(X, A; G)$ as $\operatorname{Hom}(C_n(X, A), G)$. Dualizing the short exact sequence of singular chain complexes above, we obtain a short exact sequence of cochain complexes

$$0 \to C^*(X, A; G) \xrightarrow{j^*} C^*(X; G) \xrightarrow{i^*} C^*(A; G) \to 0,$$

since dualizing exact sequences of free abelian groups preserves exactness. The **long exact sequence of cohomology groups** reads

$$\cdots \xrightarrow{\delta} H^n(X, A; G) \xrightarrow{j^*} H^n(X; G) \xrightarrow{i^*} H^n(A; G) \xrightarrow{\delta} H^{n+1}(X, A; G) \xrightarrow{j^*} \cdots .$$

We note that since exactness of the sequence $0 \to C^n(X, A; G) \xrightarrow{j^*} C^n(X; G) \xrightarrow{i^*} C^n(A; G) \to 0$ implies that j^* is injective, we may regard the group $C^n(X, A; G)$ as the subgroup of $C^n(X; G)$ consisting of cochains that vanish on chains contained in A.

For a triple $B \subset A \subset X$ of topological spaces, we similarly obtain a short exact sequence of chain complexes

$$0 \to C^*(X, A; G) \xrightarrow{j^*} C^*(X, B; G) \xrightarrow{i^*} C^*(A, B; G) \to 0$$

and the corresponding **long exact sequence of a triple**

$$\cdots \xrightarrow{\delta} H^n(X, A; G) \xrightarrow{j^*} H^n(X, B; G) \xrightarrow{i^*} H^n(A, B; G) \xrightarrow{\delta} H^{n+1}(X, A; G) \xrightarrow{j^*} \cdots .$$

2.2.6 Induced Homomorphisms

Given a continuous map $f : X \to Y$, we obtain a homomorphism of chain groups $f_\sharp : C_n(X) \to C_n(Y)$ by defining

$$f_\sharp \sigma = f \circ \sigma : \Delta^n \to Y$$

and extending linearly. Since $f_\sharp$ commutes with the boundary homomorphism ∂, we have a chain map $f_\sharp : C_*(X) \to C_*(Y)$, and a corresponding **induced homomorphism** in homology $f_* : H_n(X) \to H_n(Y)$. The dual of $f_\sharp$ is the homomorphism $f^\sharp : C^n(Y; G) \to C^n(X; G)$ which commutes with the coboundary homomorphism δ and thus induces a homomorphism $f^* : H^n(Y; G) \to H^n(X; G)$. Induced homomorphisms in homology clearly satisfy $(f \circ g)_* = f_* g_*$ and $\mathrm{id}_* = \mathrm{id}$, and similarly in cohomology we have $(f \circ g)^* = g^* f^*$ and $\mathrm{id}^* = \mathrm{id}$. These relations make homology into a covariant functor and cohomology into a contravariant functor from the category of topological spaces and continuous maps to the category of abelian groups and group homomorphisms.

Homology and cohomology groups are examples of **homotopy invariants**:

Proposition 2.2.6. *Homotopic maps* $f \simeq g : X \to Y$ *induce the same homomorphism* $f_* = g_* : H_n(X) \to H_n(Y)$ *in homology and* $f^* = g^* : H^n(Y) \to H^n(X)$ *in cohomology for all* n.

The proof is based on dividing the product $\Delta^n \times I$ into a union of simplices and defining a so-called prism operator $P : C_n(X) \to C_{n+1}(Y)$, producing a chain homotopy. Combining this theorem with the fact that homology and cohomology are functors, we obtain the following.

Corollary 2.2.7. *A homotopy equivalence induces an isomorphism of homology groups and of cohomology groups.*

An analogous result for relative homology and cohomology can be formulated using maps of pairs. A **map of a pair** $f : (X, A) \to (Y, B)$ is a continuous map $f : X \to Y$ such that $f(A) \subset B$. Such maps induce homomorphisms in relative homology and cohomology in the same way as in the absolute case. We have the following.

Proposition 2.2.8. *If two maps of pairs* $f_0, f_1 : (X, A) \to (Y, B)$ *are homotopic through maps* $f_t : (X, A) \to (Y, B)$, *then they induce the same homomorphism in relative homology and cohomology.*

The following result can be phrased by saying that singular homology is **compactly supported**. It is a consequence of the fact that every chain in $C_n(X; G)$ is contained in $C_n(K; G)$ for some compact subset $K \subset X$.

Proposition 2.2.9. *Assume that* $\{A_i\}_{i \in I}$ *is a collection of subsets of* X *such that every compact subset of* X *is contained in some* A_i. *Then the natural map*

$$\varinjlim H_n(A_i; G) \to H_n(X; G)$$

induced by the inclusions $A_i \hookrightarrow X$ *is an isomorphism.*

2.2.7 Excision

Excision is a fundamental property of relative homology and cohomology groups. If relative homology groups $H_n(X, A)$ were to describe "homology of X modulo A", we would expect that removing a nice enough set inside A would not alter the homology group $H_n(X, A)$. The precise statement is as follows.

Theorem 2.2.10 (Excision theorem)**.** *Let* $Z \subset A \subset X$ *be topological spaces such that the closure of* Z *is contained in the interior of* A. *Then the inclusion of pairs* $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ *induces an isomorphism of homology groups* $H_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} H_n(X, A)$ *and of cohomology groups* $H^n(X, A; G) \xrightarrow{\cong} H^n(X \setminus Z, A \setminus Z; G)$ *for all* n.

The theorem is proved using a process called barycentric subdivision. For each map $\sigma : \Delta^n \to X$, the n-simplex Δ^n is divided into a chain of small enough n-simplices so that the image of each small simplex is contained inside A or $X \setminus Z$. This produces a chain homotopy, yielding the desired isomorphisms in homology and cohomology.

An equivalent formulation of the theorem is obtained by setting $B = X \setminus Z$. The theorem then reads that if the interiors of sets A and B cover X, then the inclusion $(B, A \cap B) \hookrightarrow (A \cup B, A)$ induces corresponding isomorphisms in homology and cohomology. In fact, if we denote by $C_n(A + B)$ the subgroup of $C_n(X)$ generated by maps $\sigma : \Delta^n \to A \cup B$ whose image is contained in A or B, in the course of the proof of the excision theorem an isomorphism of homology groups $H_n(A + B)$ and $H_n(A \cup B)$ is established.

2.2.8 Mayer-Vietoris Sequence

In addition to the long exact sequence of relative homology and cohomology groups and the excision theorem, another indispensable tool in the study of homology and cohomology is provided by the **Mayer-Vietoris sequence**. As above, let $A, B \subset X$, and let $C_n(A + B)$ denote the subgroup of $C_n(X)$ generated by maps $\sigma : \Delta^n \to A \cup B$ whose image is contained in A or B. Then we obtain a short exact sequence

$$0 \to C_n(A \cap B) \xrightarrow{\phi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A + B) \to 0,$$

where the two middle maps are defined by $\phi(x) = (x, -x)$, and $\psi(x, y) = x + y$. Since both of these maps commute with the boundary map, the sequence extends to a short exact sequence of chain complexes

$$0 \to C_*(A \cap B) \xrightarrow{\phi} C_*(A) \oplus C_*(B) \xrightarrow{\psi} C_*(A + B) \to 0.$$

Using the fact that $H_n(A + B)$ is isomorphic to $H_n(A \cup B)$ under the assumption that the interiors of A and B cover $A \cup B$, the short exact sequence of chain complexes induces a long exact sequence in homology:

$$\cdots \xrightarrow{\partial} H_n(A \cap B) \xrightarrow{\phi_*} H_n(A) \oplus H_n(B) \xrightarrow{\psi_*} H_n(A \cup B) \xrightarrow{\partial} H_{n-1}(A \cap B) \xrightarrow{\phi_*} \cdots.$$

The connecting homomorphism can be described as follows. Using barycentric subdivision, a class $\alpha \in H_n(A \cup B)$ can be represented by a sum $x + y$ of chains contained in A and in B, respectively. Since $\partial(x + y) = 0$, the boundary $\partial x = -\partial y$ is contained in $A \cap B$. Now $\partial\alpha$ is represented by the element $\partial x = -\partial y$.

The corresponding sequence in cohomology is

$$\cdots \xrightarrow{\delta} H^n(A \cup B) \xrightarrow{\psi^*} H^n(A) \oplus H^n(B) \xrightarrow{\phi_*} H^n(A \cap B) \xrightarrow{\delta} H^{n+1}(A \cup B) \xrightarrow{\psi^*} \cdots.$$

Relative versions of the Mayer-Vietoris sequence in both homology and cohomology are obtained by considering pairs $C \subset A$ and $D \subset B$ such that the interiors of A and B cover $X = A \cup B$ and similarly the interiors of C and D cover $Y = C \cup D$. We then obtain the long exact sequence

$$\cdots \xrightarrow{\partial} H_n(A \cap B, C \cap D) \xrightarrow{\phi_*} H_n(A, C) \oplus H_n(B, D) \xrightarrow{\psi_*} H_n(X, Y) \xrightarrow{\partial} H_{n-1}(A \cap B, C \cap D) \xrightarrow{\phi_*} \cdots$$

in homology, and the corresponding long exact sequence

$$\cdots \xrightarrow{\delta} H^n(X, Y) \xrightarrow{\psi^*} H^n(A, C) \oplus H^n(B, D) \xrightarrow{\phi_*} H^n(A \cap B, C \cap D) \xrightarrow{\delta} H^{n+1}(X, Y) \xrightarrow{\psi^*} \cdots$$

in cohomology.

2.2.9 Homology of Spheres

In this section we will compute homology and cohomology groups of a few important spaces. Let us first investigate the simplest possible non-empty space, namely a point.

Proposition 2.2.11. *Let X be a one-point space. Then $H_0(X) \cong \mathbb{Z}$ and $H_n(X) = 0$ for $n \geq 1$.*

Proof. Since for each $n \geq 0$ there is a unique map $\sigma_n : \Delta^n \to X$, the chain groups $C_n(X)$ are isomorphic to $\mathbb{Z}$, with generator σ_n. The boundary of the generator is then

$$\partial\sigma_n = \sum_{i=0}^{n} (-1)^i \sigma|[v_0, ..., \hat{v}_i, ..., v_n] = \sum_{i=0}^{n} (-1)^i \sigma_{n-1} = \begin{cases} \sigma_{n-1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Hence, the chain complex has the form

$$\cdots \xrightarrow{0} C_4(X) \xrightarrow{\cong} C_3(X) \xrightarrow{0} C_2(X) \xrightarrow{\cong} C_1(X) \xrightarrow{0} C_0(X) \to 0,$$

where the map from an odd-dimensional chain group to an even-dimensional one is zero, and an isomorphism from an even-dimensional to an odd-dimensional, except at $C_0(X)$. The homology groups are clearly as stated. □

It follows from the universal coefficient theorem that the cohomology groups of a point have the same description: $H^0(X) \cong \mathbb{Z}$ and $H^n(X) = 0$ for $n \geq 1$. By homotopy invariance of homology and cohomology, spaces homotopy equivalent to a point also have these homology and cohomology groups. These spaces are called **contractible**, and important examples include Euclidean spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ and their convex subsets. In particular, the standard simplex Δ^n is contractible.

Regardless of homotopy type, non-empty and path-connected spaces have the homology group $H_0(X)$ isomorphic to $\mathbb{Z}$. This can be proved by defining a map $\epsilon : C_0(X) \to \mathbb{Z}$ by $\epsilon(\sum n_i \sigma_i) = \sum n_i$ and showing that $\operatorname{im} \partial_1 = \ker \epsilon$. This then induces an isomorphism

$$\mathbb{Z} = \operatorname{im} \epsilon \cong C_0 / \ker \epsilon = \ker \partial_0 / \operatorname{im} \partial_1 = H_0(X).$$

According to the next result, for any space X, the homology group $H_0(X)$ is a direct sum of copies of $\mathbb{Z}$, one for each path-component of X.

Proposition 2.2.12. *If X is the disjoint union of path components X_α, then the homology groups of X split as direct sums $H_n(X) = \oplus_\alpha H_n(X_\alpha)$.*

Proof. Since the image of Δ^n is contained in some path-component of X, the chain groups split as direct sums $C_n(X) = \oplus_\alpha C_n(X_\alpha)$, and since the boundary maps preserve this splitting, the homology groups also split. □

The main aim of this section is to compute the homology and cohomology groups of the spheres S^n. We will achieve this using the suspension operation. The **suspension** of a topological space X is the quotient space

$$SX = X \times I / \{ (x, 0) \sim (y, 0), (x, 1) \sim (y, 1) \ \forall\, x, y \in X \},$$

where I is the unit interval $[0, 1] \subset \mathbb{R}$. In other words, SX is the quotient of the "cylinder" $X \times I$, where the "top" $X \times \{1\}$ and the "bottom" $X \times \{0\}$ are identified separately to points. Suspension has the property of shifting homology up one dimension. The precise statement is as follows.

Proposition 2.2.13. *For any space X, we have $H_{n+1}(SX) \cong H_n(X)$ for $n \geq 1$. In addition, $H_0(X) \cong H_1(SX) \oplus \mathbb{Z}$, and $H_0(SX) \cong \mathbb{Z}$.*

Proof. Denote the collapsed points $X \times \{0\}$ and $X \times \{1\}$ by p_0 and p_1, respectively. The last isomorphism follows from the fact that SX is path-connected, since each point can be connected to either of the points p_0 and p_1. For the other isomorphisms, we will use a Mayer-Vietoris sequence. Let $U = SX \setminus \{p_0\}$ and $V = SX \setminus \{p_1\}$. Both U and V are open, and clearly $U \cup V = SX$ and $U \cap V = X \times (0, 1)$. Both U and V are contractible, since we can deformation retract each set linearly along the copies of I to the end point p_0 or p_1. In addition, $U \cap V$ has the homotopy type of X, since it deformation retracts onto $X \times \{\frac{1}{2}\}$.

The Mayer-Vietoris sequence corresponding to U and V now gives

$$\cdots \to H_{n+1}(U) \oplus H_{n+1}(V) \to H_{n+1}(SX) \to H_n(U \cap V) \to H_n(U) \oplus H_n(V) \to \cdots$$

for $n \geq 1$. Using the facts that $H_k(U) = H_k(V) = 0$ and $H_k(U \cap V) \cong H_k(X)$ for all k, the sequence splits into fractions $0 \to H_{n+1}(SX) \to H_n(X) \to 0$, which implies that $H_{n+1}(SX) \cong H_n(X)$. The last section of the Mayer-Vietoris sequence reads

$$0 \to H_1(SX) \to H_0(X) \to H_0(U) \oplus H_0(V) \to H_0(SX) \to 0.$$

Since U, V and SX are path-connected, using Proposition 2.2.12 we can write this as

$$0 \to H_1(SX) \to \bigoplus_\alpha \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0,$$

where the direct sum is over the connected components X_α of X. Since the middle map is induced by the inclusion of X to U and to V, it is easy to see that it maps the sequence $(n_\alpha)_\alpha$ to the pair $(\sum_\alpha n_\alpha, \sum_\alpha n_\alpha)$. Thus, the image of the middle map is isomorphic to $\mathbb{Z}$ and the kernel then has one summand less than $\oplus_\alpha \mathbb{Z}$. Since $H_1(SX)$ embeds in $H_0(X)$ as the kernel of the middle map, it now follows that

$$H_0(X) \cong H_1(SX) \oplus \mathbb{Z}.$$

□

To apply this result to spheres, we make the observation that S^n is homeomorphic to the suspension SS^{n-1} for all $n \geq 1$, the space S^0 being the disjoint union of two points. An explicit homeomorphism can be given by regarding S^{n-1} as the unit circle of $\mathbb{R}^n$ inside $\mathbb{R}^{n+1}$, and the cylinder $X \times [-1, 1]$ being stretched in the direction perpendicular to $\mathbb{R}^n$. The homeomorphism is then the quotient map obtained by projecting each point of $X \times [-1, 1]$ to S^n in the direction of $\mathbb{R}^n$.

From this description, we can compute the homology groups of spheres.

Theorem 2.2.14. *Let $n \geq 1$. Then $H_0(S^n) \cong H_n(S^n) \cong \mathbb{Z}$, and $H_k(S^n) = 0$ for $k \neq 0, n$. The same description holds for the cohomology groups $H^k(S^n)$.*

Proof. Since S^0 is the disjoint union of two points, we have $H_0(S^0) \cong \mathbb{Z}^2$, and $H_k(S^0) = 0$ for all other values of k. Assume first that $k > n$. By repeatedly using Proposition 2.2.13, we have

$$H_k(S^n) \cong H_{k-1}(S^{k-1}) \cong \cdots \cong H_{k-n}(S^0) = 0.$$

Next, if $k < n$, we have

$$H_k(S^n) \cong H_{k-1}(S^{k-1}) \cong \cdots \cong H_1(S^{n-k+1}),$$

and

$$H_1(S^{n-k+1}) \oplus \mathbb{Z} \cong H_0(S^{n-k}) \cong \mathbb{Z}.$$

Thus, $H_k(S^n) = 0$. Finally,

$$H_1(S^1) \oplus \mathbb{Z} \cong H_0(S^0) \cong \mathbb{Z}^2,$$

so we must have $H_1(S^1) \cong \mathbb{Z}$, and

$$H_n(S^n) \cong H_{n-1}(S^{n-1}) \cong \cdots \cong H_1(S^1) \cong \mathbb{Z}.$$

The statement about cohomology groups follows immediately from the universal coefficient theorem. □

As a final calculation in this section, we will compute the relative homology and cohomology groups of the pair $(\mathbb{R}^n, \mathbb{R}^n_0)$, where $\mathbb{R}^n_0$ denotes the set of nonzero vectors in $\mathbb{R}^n$. Since $\mathbb{R}^n$ is contractible, we have $H_k(\mathbb{R}^n) = 0$ for $k \geq 1$, and $H_0(\mathbb{R}^n) \cong \mathbb{Z}$. The space $\mathbb{R}^n_0$ in turn has the homotopy type of the sphere S^{n-1}, a deformation retraction given for example by radial projection onto the unit sphere. We thus have $H_{n-1}(\mathbb{R}^n_0) \cong H_0(\mathbb{R}^n_0) \cong \mathbb{Z}$, and $H_k(\mathbb{R}^n_0) = 0$ for $k \neq 0, n-1$.

Corollary 2.2.15. $H_n(\mathbb{R}^n, \mathbb{R}^n_0) \cong \mathbb{Z}$ *and* $H_k(\mathbb{R}^n, \mathbb{R}^n_0) = 0$ *for* $k \neq n$. *The same description holds for cohomology groups.*

Proof. In the long exact sequence of homology for the pair $(\mathbb{R}^n, \mathbb{R}^n_0)$, we have portions

$$H_k(\mathbb{R}^n) \to H_k(\mathbb{R}^n, \mathbb{R}^n_0) \to H_{k-1}(\mathbb{R}^n_0) \to H_{k-1}(\mathbb{R}^n).$$

For $k > 1$, the first and last groups are zero, so the middle map is an isomorphism

$$H_k(\mathbb{R}^n, \mathbb{R}^n_0) \cong H_{k-1}(\mathbb{R}^n_0).$$

The end of the long exact sequence reads

$$0 \to H_1(\mathbb{R}^n, \mathbb{R}^n_0) \to H_0(\mathbb{R}^n_0) \to H_0(\mathbb{R}^n) \to H_0(\mathbb{R}^n, \mathbb{R}^n_0) \to 0,$$

where the initial zero is $H_1(\mathbb{R}^n)$. If $n \geq 2$, the inclusion $\mathbb{R}^n_0 \hookrightarrow \mathbb{R}^n$ induces an isomorphism

$$H_0(\mathbb{R}^n_0) \cong H_0(\mathbb{R}^n),$$

and exactness then implies that $H_1(\mathbb{R}^n, \mathbb{R}^n_0) = H_0(\mathbb{R}^n, \mathbb{R}^n_0) = 0$.

For $n = 1$, the group $H_0(\mathbb{R}_0)$ is isomorphic to $\mathbb{Z}^2$, since $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$ has two connected components. The inclusion $\mathbb{R}_0 \hookrightarrow \mathbb{R}$ induces a surjection $H_0(\mathbb{R}_0) \to H_0(\mathbb{R})$. We thus have the exact sequence

$$0 \to H_1(\mathbb{R}, \mathbb{R}_0) \to \mathbb{Z}^2 \to \mathbb{Z} \to H_0(\mathbb{R}, \mathbb{R}_0) \to 0.$$

The kernel of the map $\mathbb{Z} \to H_0(\mathbb{R}, \mathbb{R}_0)$ is $\mathbb{Z}$, so we have $H_0(\mathbb{R}, \mathbb{R}_0) = 0$. The map $H_1(\mathbb{R}, \mathbb{R}_0) \to \mathbb{Z}^2$ has kernel equal to zero and image equal to $\mathbb{Z}$, so $H_1(\mathbb{R}, \mathbb{R}_0) \cong \mathbb{Z}$.

Again, the statement for cohomology groups follow from the universal coefficient theorem. □

It is not difficult to see that a generator of the group $H_n(\mathbb{R}^n, \mathbb{R}^n_0)$ is represented by the inclusion $\Delta^n \hookrightarrow \mathbb{R}^n$, where Δ^n is any n-simplex containing the origin in its interior. From this it follows that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a reflection, then the induced homomorphism $f_* : H_n(\mathbb{R}^n, \mathbb{R}^n_0) \to H_n(\mathbb{R}^n, \mathbb{R}^n_0)$ is multiplication by -1. It is partly based on this property that a choice of generator for $H_n(\mathbb{R}^n, \mathbb{R}^n_0)$ can be used to define an orientation for $\mathbb{R}^n$, as will be discussed later. Again, similar remarks hold for the cohomology group $H^n(\mathbb{R}^n, \mathbb{R}^n_0; G)$.

2.2.10 Cellular Cohomology

There is a powerful technique for calculating homology and cohomology groups for CW complexes, called cellular homology and cellular cohomology, respectively. The theories are completely analogous, so we will only discuss cellular cohomology. Let X be a CW complex, and recall that its n-skeleton X^n is the union of all the cells in X of dimension at most n. Cellular cohomology states that the cohomology groups of the chain complex

$$\cdots \to H^{n-1}(X^{n-1}, X^{n-2}; G) \xrightarrow{d_{n-1}} H^n(X^n, X^{n-1}; G) \xrightarrow{d_n} H^{n+1}(X^{n+1}, X^n; G) \to \cdots$$

are isomorphic to the singular cohomology groups $H^n(X; G)$. Here the the cellular boundary map d_n is the composition $\delta_n j_n$, where

$$\delta_n : H^n(X^n) \to H^{n+1}(X^{n+1}, X^n; G)$$

and

$$j_n : H^n(X^n, X^{n-1}; G) \to H^n(X^n)$$

arise from the long exact sequences of the pairs (X^{n+1}, X^n) and (X^n, X^{n-1}), respectively. Since the quotient space X^n/X^{n-1} is homeomorphic to a wedge sum of spheres S^n, one for each n-cell of X, the map d_n can be given a concrete geometric interpretation in terms of the concept of degree of a map $S^n \to S^n$. However, we will not need cellular cohomology in its full power, but merely the following two facts about cohomology of CW complexes. Firstly, $H^k(X^n; G) = 0$ if $k > n$, so in particular $H^k(X; G) = 0$ if $k > \dim X$. Secondly, the inclusion $X^n \hookrightarrow X$ induces an isomorphism $H^k(X; G) \to H^k(X^n; G)$ if $k < n$.

2.2.11 Products in Cohomology

In the definition of the cochain groups, if we take the coefficient group to be a commutative ring R, we can define an operation called **cup product** in cohomology using the multiplication of R. On the level of cochains, this is defined as follows. Let $\phi \in C^k(X;R)$ and $\psi \in C^l(X;R)$. Define $\phi \smile \psi \in \mathbb{C}^{k+l}(X;R)$ to be the cochain that satisfies

$$(\phi \smile \psi)(\sigma) = \phi(\sigma|[\nu_0, ..., \nu_k])\psi(\sigma|[\nu_k, ..., \nu_{k+l}]).$$

The relation $\delta(\phi \smile \psi) = \delta\phi \smile \psi + (-1)^k \phi \smile \delta\psi$, which is verified with a direct calculation, shows that there is a well-defined induced product

$$\smile : H^k(X;R) \times H^l(X;R) \longrightarrow H^{k+l}(X;R).$$

This product is associative and distributive, and if R has an identity, then the element $\mathbb{1} \in H^0(X;R)$, represented by $C_0(X) \to R, x \mapsto 1$, defines an identity for the cup product. Using the cup product, we can make the direct sum of the cohomology groups of X into a graded ring

$$H^*(X;R) = \oplus_{k=0}^{\infty} H^k(X;R),$$

the graded pieces being the cohomology groups of different dimensions.

A relative cup product can be defined as follows. Let $A, B \subset X$ be two open sets. Denote by

$$C^n(X, A+B; R)$$

the subgroup of $C^n(X;R)$ of cochains vanishing on sums of chains contained in A and in B. As mentioned in the discussion of excision, the inclusion $C^n(X, A \cup B; R) \to C^n(X, A+B; R)$ induces an isomorphism $H^n(X, A \cup B; R) \xrightarrow{\cong} H^n(X, A+B; R)$ for all n. Now, if $\phi \in C^l(X, A; R)$ and $\psi \in C^l(X, B; R)$, then $\phi \smile \psi$ vanishes on chains contained in both A and B, in other words $\phi \smile \psi \in C^{k+l}(X, A+B; R)$. This induces a cup product in cohomology, and composing with the previous isomorphism we obtain the relative cup product

$$H^k(X, A; R) \times H^l(X, B; R) \longrightarrow H^{k+l}(X, A \cup B; R).$$

Using the cup product, we can now define the **cross product** operation, relating the cohomology groups of two spaces X and Y with the cohomology groups of their product $X \times Y$. The projections

$$pr_X : X \times Y \to X \quad \text{and} \quad pr_Y : X \times Y \to Y$$

induce maps

$$pr_X^* : H^n(X;R) \to H^n(X \times Y; R),$$
$$pr_Y^* : H^n(Y;R) \to H^n(X \times Y; R),$$

so we define the cross product as the map

$$\times \; : H^k(X;R) \times H^l(Y;R) \longrightarrow H^{k+l}(X \times Y; R)$$

taking (ϕ, ψ) to $\phi \times \psi = pr_X^*(\phi) \smile pr_Y^*(\psi)$. It follows from the corresponding properties of the cup product that the cross product is associative and distributive as well. A relative version for pairs (X, A) and (Y, B) is defined identically and has the form

$$\times \; : H^k(X, A; R) \times H^l(Y, B; R) \longrightarrow H^{k+l}(X \times Y, A \times Y \cup X \times B; R)$$

There is one more form of product, called the **cap product**, that we will use later. This is defined as the bilinear pairing

$$\frown : C_k(X, A; R) \times C^l(X, A; R) \to C_{k-l}(X; R)$$

given by the formula

$$\sigma \frown \phi = \phi(\sigma|[v_0, ..., v_l])\sigma|[v_l, ..., v_k].$$

If $l > k$, we define $\sigma \frown \phi$ to be zero. It is easy to see that $\sigma \frown \phi$ is the unique element such that for all $\psi \in C^{k-l}(X)$,

$$\psi(\phi \frown \sigma) = (\phi \smile \psi)(\sigma).$$

From this it is straightforward to derive the formulas $(\phi \smile \psi) \frown \sigma = \phi \frown (\psi \frown \sigma)$ and $1 \frown \sigma = \sigma$. Furthermore, it follows from the identity

$$\partial(\sigma \frown \phi) = (-1)^l(\partial\sigma \frown \phi - \sigma \frown \delta\phi)$$

that the cap product induces a corresponding operation

$$\frown : H_k(X, A; R) \times H^l(X, A; R) \to H_{k-l}(X; R)$$

on homology and cohomology groups.

We will end this section by stating a result concerning cohomology of a product space of CW complexes. One would hope that there is a simple relationship between the cohomology rings $H^*(X; R)$ and $H^*(Y; R)$ and the ring $H^*(X \times Y; R)$. In favorable cases such a relationship exists, and it is given in terms tensor product and the cross product. The cross product defined above defines a bilinear map from $H^k(X; R) \times H^l(X; R)$ to $H^{k+l}(X; R)$, so by the definition of tensor product it extends into a homomorphism

$$H^k(X; R) \otimes H^l(X; R) \longrightarrow H^{k+l}(X; R).$$

Theorem 2.2.16 (Künneth formula)**.** *If X and Y are CW complexes, and if $H^k(Y; R)$ is a finitely generated free R-module for all k, then the cross product $H^*(X; R) \otimes H^*(Y; R) \longrightarrow H^*(X \times Y; R)$ is a ring isomorphism.*

Chapter 3

The Grassmannian

In this section we define the complex Grassmannian $G_n(\mathbb{C}^k)$ and the infinite Grassmannian G_n, and prove their basic properties.

3.1 Definitions and Basic Properties

Let $\mathbb{C}^k$ be the k-dimensional complex vector space endowed with the Hermitian inner product. We start by defining the finite Grassmannian as a set.

Definition 3.1.1. *Let n and k be natural numbers with $k \geq n$. The **Grassmannian** $G_n(\mathbb{C}^k)$ is the set of n-dimensional subspaces of the vector space $\mathbb{C}^k$.*

We obtain an important special case of $G_n(\mathbb{C}^k)$ by setting $n = 1$. The space $G_1(\mathbb{C}^k)$ is called the **complex projective space**, and is denoted by $\mathbb{CP}^{k-1}$. In this case, $k-1$ is the dimension of the projective space as a complex analytic space.

Our first goal is to endow the Grassmannian with the structure of a compact topological manifold. As a first step, we will define a topology on $G_n(\mathbb{C}^k)$ using an auxiliary space.

Definition 3.1.2. *An **orthonormal n-frame** is an n-tuple $(v_1, ..., v_n)$ of vectors in $\mathbb{C}^k$ such that $\{v_1, ...v_n\}$ is an orthonormal set. The **Stiefel manifold** $V_n(\mathbb{C}^k)$ is the set of all orthonormal k-frames.*

The Stiefel manifold is topologized with the subspace topology inherited from the n-fold product of the unit sphere in $\mathbb{C}^k$. There is a canonical surjection

$$q : V_n(\mathbb{C}^k) \longrightarrow G_n(\mathbb{C}^k)$$

sending the n-frame $(v_1, ..., v_n)$ to the subspace with basis $\{v_1, ...v_n\}$, and the Grassmannian is endowed with the quotient topology induced by this map. This by definition makes q into a continuous map.

There is a variant of the Stiefel manifold defined above, which we denote by $\widetilde{V}_n(\mathbb{C}^k)$. This is defined as the set of linearly independent n-tuples of vectors in $\mathbb{C}^k$ and given the subspace topology from $\mathbb{C}^{nk}$. There is again a canonical surjection

$$\tilde{q} : \widetilde{V}_n(\mathbb{C}^k) \to G_n(\mathbb{C}^k),$$

and we can give $G_n(\mathbb{C}^k)$ the topology induced by $\tilde{q}$. This topology coincides with the one defined in the previous paragraph, since the following diagram commutes.

$$\begin{tikzcd}
V_n(\mathbb{C}^k) \arrow[r, hook] \arrow[dr, "q"'] & \widetilde{V}_n(\mathbb{C}^k) \arrow[r] \arrow[d, "\tilde{q}"] & V_n(\mathbb{C}^k) \arrow[dl, "q"] \\
& G_n(\mathbb{C}^k) &
\end{tikzcd}$$

Here the top left map is the inclusion and the top right map is defined by performing the Gram-Schmidt process. We note that $\widetilde{V}_n(\mathbb{C}^k)$ is an open set of $(\mathbb{C}^k)^n$. This can be seen as follows. Points in $\widetilde{V}_n(\mathbb{C}^k)$ can be represented by $k \times n$ complex matrices

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix}.$$

The rows of A are linearly independent if and only if at least one $n \times n$ minors have nonzero determinant. Now if $M^{k\times n}(\mathbb{C})$ denotes the set of all $k \times n$ complex matrices, there is a continuous map $M^{k\times n}(\mathbb{C}) \to \mathbb{C}^{\binom{k}{n}}$ given by taking the determinant of each of the $n \times n$ minors of A to each of the coordinates. Then $\widetilde{V}_n(\mathbb{C}^k)$ is the preimage of $\mathbb{C}^{\binom{k}{n}} \setminus \{0\}$ under this map, hence open.

We will now prove the following theorem that lists some topological properties of the Grassmannian.

Theorem 3.1.3. *The Grassmannian $G_n(\mathbb{C}^k)$ is a compact, path-connected, topological manifold of dimension $2n(k-n)$.*

Proof. We will first show that $G_n(\mathbb{C}^k)$ is a Hausdorff space. Let $w \in \mathbb{C}^n$, and define the function

$$\rho_w : G_n(\mathbb{C}^k) \to \mathbb{R}, \qquad \rho_w(X) = \min\{\, \|x - w\| \mid x \in X\}.$$

If $\{x_1, ..., x_n\}$ is an orthonormal basis for X, then the formula

$$\rho_w(X) = w \cdot w - \sum_{i=1}^{n} (w \cdot x_i)^2$$

shows that the composition $\rho_w \circ q$ is continuous, and hence ρ_w is. Let now X and Y be two distinct elements in $G_n(\mathbb{C}^k)$, and let $w \in \mathbb{C}^k$ be a point such that $w \in X$ and $w \notin Y$. Then $\rho_w(X) = 0$ but $\rho_w(Y) \neq 0$. Thus any two points of $G_n(\mathbb{C}^k)$ can be separated by a continuous function, so $G_n(\mathbb{C}^k)$ is Hausdorff.

Next we will construct a Euclidean neighborhood of real dimension $2n(k-n)$ around an arbitrary point of $G_n(\mathbb{C}^k)$ using the following strategy. Let $X \in G_n(\mathbb{C}^k)$ be a point, and consider $\mathbb{C}^k$ as the direct sum $X \oplus X^\perp$, where $X^\perp$ is the orthogonal complement of X. Define the set

$$U_X = \{Y \in G_n(\mathbb{C}^k) \mid Y \cap X^\perp = 0\}.$$

We will show that U_X is homeomorphic to $\mathrm{Hom}(X, X^\perp)$, which in turn is homeomorphic to $\mathbb{C}^{n(k-n)}$, as we can identify it with the set of complex $(k-n) \times n$-matrices.

The set U_X is open, since if v is any basis vector for any subspace $Y \in U_X$, the projection of v onto X is nonzero, and thus there is an open neighborhood around v with no vectors in $X^\perp$. For $Y \in U_X$, denote by p_Y the projection map $pr_X : X \oplus X^\perp \to X$ restricted to Y. The definition of U_X is equivalent to requiring that p_Y is a surjection, and hence a linear isomorphism, so that there exists an inverse $p_Y^{-1} : X \to Y$. Now define the linear map $T_Y : X \to X^\perp$ as the composition

$$X \xrightarrow{p_Y^{-1}} X \oplus X^\perp \xrightarrow{pr_{X^\perp}} X^\perp.$$

The subspace Y can now be described as the graph of T_Y, that is,

$$Y = \{ (x, T_Y(x)) \in X \oplus X^\perp \mid x \in X \}.$$

This gives us a correspondence $T : U_X \to \mathrm{Hom}(X, X^\perp)$ taking the subspace Y to the linear map T_Y. This correspondence is bijective, with the inverse T^{-1} given by taking the graph. Since X has complex dimension n and $X^\perp$ has complex dimension $k-n$, the set $\mathrm{Hom}(X, X^\perp)$, considered as matrices, is homeomorphic to $\mathbb{C}^{n(k-n)}$, or to $\mathbb{R}^{2n(k-n)}$. It remains to show that both T and T^{-1} are continuous.

Let $\{x_1, ..., x_n\}$ be a fixed orthonormal basis of X. Since T_Y is bijective for every $Y \in U_X$, there exists a unique basis $\{y_1, ..., y_n\}$ of Y such that $p_Y(y_i) = x_i$ for $i = 1, ..., n$. The map $U_X \to \mathbb{C}^k$ sending $Y \in G_n(\mathbb{C}^k)$ to y_i is continuous if and only if the corresponding map $\widetilde{V}_n(\mathbb{C}^k) \supset \tilde{q}^{-1}(U_X) \to \mathbb{C}^k$ is continuous. But this map, given by the projection of x_i onto Y in the direction of $X^\perp$, can be written down explicitly with formulas depending continuously on the coordinates of the chosen basis vectors of Y, which constitute a point in $\tilde{q}^{-1}(U_X)$. It now follows from the identity $y_i = x_i + T_Y(x_i)$ that $T_Y(x_i)$ depends continuously on Y for all x_i, and thus the map T_Y depends continuously on Y. This shows continuity of T. On the other hand, since T_Y is given by a complex matrix, the above identity shows that y_i depends continuously on T_Y, and hence Y depends continuously on T_Y. This shows continuity of T^{-1}.

To show that the Grassmannian $G_n(\mathbb{C}^k)$ is compact, we note that the Stiefel manifold $V_n(\mathbb{C}^k)$ can be described as the set of matrices

$$V_n(\mathbb{C}^k) = \{ A \in M_{k\times n}(\mathbb{C}) \mid A^T A = I_n \},$$

where the columns of each matrix A correspond to the given orthonormal basis. Since $V_n(\mathbb{C}^k)$ is given as the common zero set of a collection of polynomials, it is closed, and it is bounded since every entry in a given matrix A has an absolute value of at most one. Thus $V_n(\mathbb{C}^k)$ is compact, and since $G_n(\mathbb{C}^k)$ is the image of the compact set $V_n(\mathbb{C}^k)$ under the continuous map q, it is itself compact.

We can now deduce that $G_n(\mathbb{C}^k)$ is second countable as follows. Since every point of $G_n(\mathbb{C}^k)$ has a Euclidean neighborhood, it can be covered by such neighborhoods, and since it is compact, already a finite number of these neighborhoods cover $G_n(\mathbb{C}^k)$. Each of these neighborhoods is second countable, so each of them has a countable basis. The union of these bases is a countable collection and forms a basis for $G_n(\mathbb{C}^k)$.

Finally, to show that $G_n(\mathbb{C}^k)$ is path-connected, we first show that $\widetilde{V}_n(\mathbb{C}^k)$ has this property. Each point in $\widetilde{V}_n(\mathbb{C}^k)$ is an n-tuple $(v_1, ..., v_n)$ of linearly independent vectors in $\mathbb{C}^k$. If $(w_1, ..., w_n)$ is another point, then there exists an invertible matrix $A \in GL_n(\mathbb{C})$ such that

$$(w_1, ..., w_n) = (Av_1, ..., Av_n).$$

By Theorem 2.1.6, there exists a path $\gamma : [0, 1] \to GL_n(\mathbb{C})$ such that $\gamma(0) = I$ and $\gamma(1) = A$. Then $\gamma' : [0, 1] \to \widetilde{V}_n(\mathbb{C}^k)$ defined by

$$\gamma'(t) = (\gamma(t)v_1, ..., \gamma(t)v_n)$$

is a path in $\widetilde{V}_n(\mathbb{C}^k)$ connecting the points $(v_1, ..., v_n)$ and $(w_1, ..., w_n)$. Thus $\widetilde{V}_n(\mathbb{C}^k)$ is path-connected, and since $G_n(\mathbb{C}^k)$ is a continuous image of $\widetilde{V}_n(\mathbb{C}^k)$, it is also path-connected. □

We will now define the infinite Grassmannian as a topological space. Denote by $\mathbb{C}^\infty$ the set of sequences of complex number with only finitely many non-zero terms, and endow $\mathbb{C}^\infty$ with both the obvious complex vector space structure and the direct limit topology arising from the sequence of inclusions

$$\mathbb{C} \subset \mathbb{C}^2 \subset \mathbb{C}^3 \subset ... \subset \mathbb{C}^m \subset ... \subset \mathbb{C}^\infty,$$

where the inclusions are given by the obvious formula $(z_1, ..., z_n) \mapsto (z_1, ..., z_n, 0)$. Define the infinite Grassmannian G_n to be the set of all n-dimensional subspaces of $\mathbb{C}^\infty$, and similarly endow it with the direct limit topology arising from the sequence of inclusions

$$G_n(\mathbb{C}^n) \subset G_n(\mathbb{C}^{n+1}) \subset G_n(\mathbb{C}^{n+2}) \subset ... \subset G_n(\mathbb{C}^{n+m}) \subset ... \subset G_n.$$

In the case $n = 1$ we get the infinite complex projective space $\mathbb{CP}^\infty$.

3.2 CW Structure for the Grassmannian

In this section we will describe a CW structure for the Grassmannian $G_n(\mathbb{C}^k)$ and the infinite Grassmannian G_n. Let $X \subset \mathbb{C}^k$ be an n-dimensional subspace, that is, a point in $G_n(\mathbb{C}^k)$. Firstly, we have

$$0 \leq \dim(X \cap \mathbb{C}) \leq \dim(X \cap \mathbb{C}^2) \leq ... \leq \dim(X \cap \mathbb{C}^k) = n,$$

where the dimensions are complex. Secondly, for $1 \leq i \leq m$, the sequence

$$0 \longrightarrow X \cap \mathbb{C}^{i-1} \hookrightarrow X \cap \mathbb{C}^i \longrightarrow \mathbb{C}$$

is exact, where the last map is the projection onto the ith coordinate. Since this last map is either the zero map or a surjection, the dimension of $X \cap \mathbb{C}^{i-1}$ and $X \cap \mathbb{C}^i$ can differ by at most one. By keeping track of when these dimensions grow, we can organize the points of the Grassmannian in a suitable way. For this purpose, we define a **Schubert symbol** to be an n-tuple $\sigma = (\sigma_1, ..., \sigma_n) \in \mathbb{N}^n$ such that $1 \leq \sigma_1 < \sigma_2 < ... < \sigma_n \leq k$, and for each such Schubert symbol σ, we define the set

$$e(\sigma) = \{ X \in G_n(\mathbb{C}^k) \mid \dim(X \cap \mathbb{C}^{\sigma_i}) = i, \dim(X \cap \mathbb{C}^{\sigma_i - 1}) = i - 1 \}.$$

Clearly, as σ varies over all possible Schubert symbols, each point $X \in G_n(\mathbb{C}^k)$ belongs to exactly one of the sets $e(\sigma)$. These sets are called **Schubert cells**, and they can be described in terms of matrices as follows. An n-plane $X \in G_n(\mathbb{C}^k)$ is in $e(\sigma)$ if and only if it is spanned by the rows of an $n \times k$ matrix of the form

$$M = \begin{pmatrix} a_{11} & \cdots & a_{1\sigma_1} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ a_{21} & \cdots & a_{2\sigma_1} & \cdots & \cdots & a_{2\sigma_2} & \cdots & 0 & \cdots & 0 & \cdots \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n\sigma_1} & \cdots & \cdots & a_{n\sigma_2} & \cdots & a_{n\sigma_k} & \cdots & 0 & \cdots \end{pmatrix},$$

where on jth row the element $a_{j\sigma_j}$ is nonzero and all the elements to the right from it are zero.

We will now prove that the Schubert cells $e(\sigma)$ are the cells of a CW complex structure on $G_n(\mathbb{C}^k)$ and describe the characteristic maps. Since there are only finitely many Schubert symbols, it suffices to produce for each Schubert symbol σ a map $D^n_\sigma \to X$ from a closed cell D^n_σ to X that carries the interior of D^n_σ homeomorphically onto $e(\sigma)$ and maps each point on the boundary of D^n_σ to a cell of lower dimension.

The characteristic maps turn out to be nothing else than the quotient map $q : V_n(\mathbb{C}^k) \to G_n(\mathbb{C}^k)$ restricted to a certain subspace. To define this subspace, we restrict attention to a particular basis for each n-plane in $\mathbb{C}^k$. Define the half-space

$$H^l = \{ (\xi_1, ..., \xi_l, 0, ..., 0) \in \mathbb{C}^k \mid \xi_l \in \mathbb{R}_+ \},$$

where $\mathbb{R}_+ = \{r \in \mathbb{R} \mid r > 0\}$. The closure of H^l is

$$\overline{H}^l = \{ (\xi_1, ..., \xi_l, 0, ..., 0) \in \mathbb{C}^k \mid \xi_l \in \mathbb{R}_+ \cup \{0\} \}.$$

We have the following result.

Lemma 3.2.1. *Each subspace* $X \in e(\sigma)$ *has a unique orthonormal basis*

$$(x_1, ..., x_n) \in H^{\sigma_1} \times ... \times H^{\sigma_n}.$$

Proof. Since $X \cap \mathbb{C}^{\sigma_1}$ has by definition one complex dimension, the conditions $\|(\xi_1, ..., \xi_{\sigma_1}, 0, ...0)\| = 1$ and $\xi_k \in \mathbb{R}_+$ specify a unique vector $x = (\xi_1, ..., \xi_k, 0, ...0) \in X \cap H^{\sigma_1}$. Let this vector be x_1. Continuing inductively, assume that we have basis vectors $x_1, ..., x_{i-1}$ with each $x_j \in H^{\sigma_j}$. The space $X \cap \mathbb{C}^{\sigma_i}$ has dimension i, so the conditions $x = (\xi_1, ..., \xi_{\sigma_i}, 0, ..., 0) \perp \{x_1, ..., x_{i-1}\}$, $\|x\| = 1$, and $\xi_k \in \mathbb{R}_+$ again specify a unique vector. Let this vector be x_i. □

Now define the sets

$$e'(\sigma) = V_n(\mathbb{C}^k) \cap (H^{\sigma_1} \times ... \times H^{\sigma_n}),$$
$$\overline{e}'(\sigma) = V_n(\mathbb{C}^k) \cap (\overline{H}^{\sigma_1} \times ... \times \overline{H}^{\sigma_n}).$$

The set $\overline{e}'(\sigma)$ will be the domain of the characteristic map of $e(\sigma)$. We first prove the following.

Lemma 3.2.2. *The set* $\overline{e}'(\sigma)$ *is a closed cell of real dimension* $d(\sigma) = 2\sum_{i=1}^{n}(\sigma_i - i)$. *The interior of* $\overline{e}'(\sigma)$ *is* $e'(\sigma)$.

Proof. First consider the case $n = 1$, so that $\sigma = (\sigma_1)$, and

$$\overline{e}'(\sigma) = \{ (\xi_1, ..., \xi_{\sigma_1}, 0, ..., 0) \in \mathbb{C}^k \mid \sum_{j=1}^{\sigma_1} |\xi_j|^2 = 1, \operatorname{Re}(\xi_{\sigma_1}) \geq 0, \operatorname{Im}(\xi_{\sigma_1}) = 0 \}.$$

This is a closed hemisphere of dimension $2\sigma_1 - 2$, which is homeomorphic to a closed disc. The interior is an open hemisphere, homeomorphic to an open disc, since $\operatorname{Re}(\xi_{\sigma_1}) > 0$.

Proceeding with induction on n, assume now that $\overline{e}'(\sigma)$ is homeomorphic to a closed disc of dimension $d(\sigma)$, where $\sigma = (\sigma_1, ..., \sigma_n)$ is a fixed Schubert symbol. Let $\sigma_{n+1} > \sigma_n$ and denote

$$\tilde{\sigma} = (\sigma_1, ..., \sigma_n, \sigma_{n+1}).$$

Denote by b_i the vector in $(0, ..., 0, 1, 0, ..., 0) \in H^{\sigma_i}$ whose σ_ith coordinate equals 1. Define the set

$$D = \{ u \in \overline{H}^{\sigma_{n+1}} \mid |u| = 1, \; b_i \cdot u = 0 \;\; \forall \, 1 \leq i \leq n \}.$$

The vectors in D have each σ_ith coordinate equal to 0 for $i \leq n$, and the rest of the coordinates parametrize a closed hemisphere of dimension $2(\sigma_{n+1} - n - 1)$. Thus, D is homeomorphic to a closed disc. The interior of D is $D \cap H^{\sigma_{n+1}}$. By the induction hypothesis, $\overline{e}'(\sigma) \times D$ is homeomorphic to a closed disc of dimension

$$d(\sigma) + 2(\sigma_{n+1} - n - 1) = 2\sum_{i=1}^{n}(\sigma_i - i) + 2(\sigma_{n+1} - n - 1) = d(\tilde{\sigma}),$$

with interior $e'(\sigma) \times \operatorname{int} D$.

We will next define a homeomorphism f between $\overline{e}'(\sigma) \times D$ and $\overline{e}'(\tilde{\sigma})$. For this purpose, given two unit vectors $u, v \in \mathbb{C}^k$ such that $u \neq -v$, define $T(u, v)$ to be the unique rotation that takes u to v and leaves all vectors orthogonal to u and v fixed. $T(u, v)$ is given by the formula

$$T(u, v)x = x - \frac{(u + v) \cdot x}{1 + u \cdot v}(u + v) + 2(u \cdot x)v.$$

This formula gives the correct map, since firstly it is linear, and secondly,

$$T(u, v)u = u - \frac{1 + u \cdot v}{1 + u \cdot v}(u + v) + 2(u \cdot u)v = u - u - v + 2v = v,$$

so $T(u,v)$ has the correct effect on u. Thirdly,

$$T(u,v)v = v - \frac{1+u\cdot v}{1+u\cdot v}(u+v) + 2(u\cdot v)v = 2(u\cdot v)v - u,$$

and hence

$$|T(u,v)v|^2 = (2(u\cdot v)v - u)\cdot(2(u\cdot v)v - u) = 4(u\cdot v)^2 - 4(u\cdot v)^2 + 1 = 1,$$

so $T(u,v)$ is a rotation in the plane spanned by u and v. Finally, if x is orthogonal to u and v, then

$$T(u,v)x = x.$$

From the formula we also note that $T(u,v)x$ is continuous as a function of u, v and x, and if $u, v \in \mathbb{C}^l \subset \mathbb{C}^k$, then $T(u,v)x - x$ is just a linear combination of x, u, and v, so in particular

$$T(u,v)x \equiv x \text{ modulo } \mathbb{C}^l.$$

By definition, $T(u,u)$ is the identity map, and $T(u,v)^{-1} = T(v,u)$.

Let now $X = (x_1, ..., x_n) \in \overline{e}'(\sigma)$ be an n-tuple of orthonormal vectors $x_i \in \overline{H}^{\sigma_i}$. Define a linear transformation $T_X : \mathbb{C}^k \to \mathbb{C}^k$ by

$$T_X = T(b_n, x_n) \circ T(b_{n-1}, x_{n-1}) \circ \cdots \circ T(b_1, x_1).$$

The map T_X carries each b_i to x_i. Namely, if $j < i$, then

$$b_i \cdot b_j = b_i \cdot x_j = 0,$$

so $T(b_j, x_j)$ fixes b_i. By definition, $T(b_i, x_i)b_i = x_i$, and if $j > i$, then

$$x_j \cdot x_i = b_j \cdot x_i = 0,$$

so $T(b_j, x_j)$ fixes x_i.

Now define the map

$$\begin{aligned} f : \overline{e}'(\sigma) \times D &\longrightarrow \overline{e}'(\tilde{\sigma}) \\ (X, u) &\longmapsto (x_1, ..., x_n, T_X u), \end{aligned}$$

where $X = (x_1, ..., x_n) \in \overline{e}'(\sigma)$. We note that since

$$T_X u \equiv u \text{ modulo } \mathbb{C}^{\sigma_n},$$

we have $T_X u \in \overline{H}^{\sigma_{n+1}}$, and if (X, u) is an interior point, then so is its image under f. Also, since T_X is a composition of rotations, it is itself a rotation. This implies that

$$x_i \cdot T_X u = T_X b_i \cdot T_X u = b_i \cdot u = 0$$

for all $1 \leq i \leq n$, and that $T_X u$ is a unit vector. Hence, $f(X,u) \in \overline{e}'(\tilde{\sigma})$, and f is well-defined. The inverse of f is given by

$$f^{-1}(x_1, ..., x_{n+1}) = ((x_1, ..., x_n), T_X^{-1} x_{n+1}),$$

where

$$T_X^{-1} = T(x_1, b_1) \circ \cdots \circ T(x_n, b_n).$$

The fact that f^{-1} is well-defined is deduced from similar remarks as above. Both $\overline{e}'(\sigma) \times D$ and $\overline{e}'(\tilde{\sigma})$ can be viewed as subsets of complex coordinate spaces, so when we consider f and f^{-1} as restrictions of maps between coordinate spaces, it follows from the formula for $T(u,v)x$ that both f and f^{-1} are continuous. We have thus shown that f is a homeomorphism. $\square$

We are now ready to describe the CW structure of the finite Grassmannian.

Theorem 3.2.3. *For every Schubert symbol* σ*, the quotient map*

$$q : V_n(\mathbb{C}^k) \longrightarrow G_n(\mathbb{C}^k)$$

takes $e'(\sigma)$ *homeomorphically onto* $e(\sigma)$*. Every point on the boundary* $\overline{e}(\sigma) \setminus e(\sigma)$ *belongs to a cell* $e(\tau)$ *of lower dimension. Thus, the Schubert cells* $e(\sigma)$ *form a CW decomposition of the Grassmannian* $G_n(\mathbb{C}^k)$*, as* σ *varies over all Schubert symbols. The characteristic map of each cell is given by the restriction of the canonical projection* $q : V_n(\mathbb{C}^k) \to G_n(\mathbb{C}^k)$ *to the set* $\overline{e}'(\sigma)$*.*

Proof. The quotient map q is by definition continuous, and by Lemma 3.2.1, it restricts to a bijection on $e'(\sigma)$. It suffices to show that the restriction is a closed map. Let $A \subset e'(\sigma)$ be a relatively closed set, that is, closed in the subspace topology of $e'(\sigma)$. Let $\overline{A}$ be the closure of A in $V_n(\mathbb{C}^k)$. Then $\overline{A} \subset \overline{e}'(\sigma)$, since $\overline{e}'(\sigma)$ is closed in $V_n(\mathbb{C}^k)$. Now, since $\overline{e}'(\sigma)$ is compact, so is $\overline{A}$, and thus $q(\overline{A}) \subset G_n(\mathbb{C}^k)$ is compact, hence closed. We have

$$q(A) = q(\overline{A} \cap e'(\sigma)) \subset q(\overline{A}) \cap q(e'(\sigma)) = q(\overline{A}) \cap e(\sigma).$$

To show the other inclusion, assume that $(x_1, ..., x_n) \in \overline{A} \setminus A$. Then $(x_1, ..., x_n) \notin e'(\sigma)$, so for some $1 \leq i \leq n$, we have $x_i \in \mathbb{C}^{\sigma_i - 1}$, so

$$\dim(q(x_1, ..., x_n) \cap \mathbb{C}^{\sigma_i - 1}) \geq i.$$

Hence $q(x_1, ..., x_n) \notin e(\sigma)$, so $q(\overline{A}) \cap e(\sigma) \subset q(A)$. Thus

$$q(A) = q(\overline{A}) \cap e(\sigma),$$

and $q(A)$ is relatively closed in $e(\sigma)$, so q restricts to a closed map on $e'(\sigma)$. This proves the first assertion.

Since $\overline{e}'(\sigma)$ is compact and $G_n(\mathbb{C}^k)$ is Hausdorff, the image $q(\overline{e}'(\sigma))$ is closed. Thus,

$$q(\overline{e}'(\sigma)) = \overline{q(\overline{e}'(\sigma))} = \overline{q(e'(\sigma))} = \overline{e}(\sigma).$$

Hence every point $X \in \overline{e}(\sigma) \setminus e(\sigma)$ has an orthonormal basis $(x_1, ..., x_n) \in \overline{e}'(\sigma)$. We have

$$\dim(X \cap \mathbb{C}^{\sigma_i}) \geq i,$$

and since $X \notin e(\sigma)$, for some i we must have $x_i \in \mathbb{C}^{\sigma_i - 1}$. Let $\tau = (\tau_1, ..., \tau_n)$ be the Schubert symbol associated to X. It now follows from the above inequality that $\tau_i \leq \sigma_i$ for $1 \leq i \leq n$, and since $X \notin e(\sigma)$, we must actually have $\tau_j < \sigma_j$ for some j. Thus the Schubert cell $e(\tau)$ containing X must have strictly lower dimension that $e(\sigma)$. □

It is now easy to describe a CW structure for the infinite Grassmannian G_n. Without bounding the indices of a Schubert symbol $\sigma = (\sigma_1, ..., \sigma_n)$ from above, we can define sets $e(\sigma) \in G_n$ just as in the case of the finite Grassmannian. As the Schubert cells vary through all possibilities, we see that the cells $e(\sigma)$ cover G_n. Since each $e(\sigma)$ is contained in some finite Grassmannian $G_n(\mathbb{C}^k) \subset G_n$, it is clear that the first two conditions in the definition of a CW complex are satisfied. To check that the third one holds, we simply observe that if a set meets each cell in a closed set, then it meets every finite Grassmannian in a closed set, so it is closed in G_n by the definition of the direct limit topology. Characteristic maps are given by restricting the projection $q : V_n(\mathbb{C}^k) \to G_n(\mathbb{C}^k)$ to $e(\sigma)$, where k is some sufficiently large integer. We have thus proved the following.

Theorem 3.2.4. *As* σ *varies over all Schubert symbols, the Schubert cells* $e(\sigma)$ *form a CW decomposition of the infinite Grassmannian* G_n*.*

Chapter 4

Vector Bundles

In this work, we focus on complex vector bundles. However, as complex vector spaces are real vector spaces with additional structure, similarly complex vector bundles are real vector bundles with additional structure. For this reason we will begin the study of vector bundles by defining real vector bundles, and then describe the additional structure required to turn a real vector bundle into a complex one.

4.1 Definition and First Properties

Definition 4.1.1. *A* ***real vector bundle of rank*** n *is a continuous map of topological spaces* $\pi : E \to B$ *such that for each* $x \in B$,

1. *the* ***fiber*** $\pi^{-1}(\{x\}) \subset E$ *has the structure of a real vector space of dimension n,*
2. *there exists an open neighborhood* $U \subset B$ *of* x *and a homeomorphism* $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ *that restricts to a linear isomorphism* $\pi^{-1}(\{y\}) \to \{y\} \times \mathbb{R}^n$ *for each* $y \in U$.

The space B *is called the* ***base space*** *and the space* E *is called the* ***total space*** *of the vector bundle, and* π *is called the* ***projection map****. A pair* (U, ϕ) *satisfying the second condition is called a* ***local trivialization****.*

We will frequently denote a vector bundle with only the total space E, leaving the rest of the data implicit. The fiber $\pi^{-1}(\{x\})$ will be sometimes denoted by F_x. Local trivializations are by no means unique. In fact, if (U, ϕ) is a local trivialization, and $g : U \to GL_n(\mathbb{R})$ is any continuous map from U to the general linear group $GL_n(\mathbb{R})$, we can define another local trivialization (U, ψ) by

$$\psi(y) = (x, g(x)v),$$

where $(x, v) = \phi(y)$ and $y \in \pi^{-1}(U)$. On the other hand, given two local trivializations (U, ϕ) and (U, ψ) over $U \subset B$, consider the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\phi"] \arrow[d, "\psi"'] & U \times \mathbb{R}^n \arrow[dl, "\psi \circ \phi^{-1}"] \\ U \times \mathbb{R}^n & \end{tikzcd}.$$

We have the map $\psi \circ \phi^{-1} : U \times \mathbb{R}^n \to U \times \mathbb{R}^n$ that takes $\{x\} \times \mathbb{R}^n$ to itself by a linear isomorphism. Now define the map $g : U \to GL_n(\mathbb{R})$ whose coordinate functions are the compositions

$$g_{ij} : U \xrightarrow{\cong} U \times \{e_j\} \xrightarrow{\psi\circ\phi^{-1}} U \times \mathbb{R}^n \xrightarrow{pr_2} \mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R} \xrightarrow{pr_i} \mathbb{R},$$

where the two last maps are the projections to the second factor and to the ith factor respectively. Since each coordinate function g_{ij} is a composition of continuous maps, the matrix valued function $g = (g_{ij})$ is continuous. The composition $\psi \circ \phi^{-1}$ is now given explicitly by $\psi \circ \phi^{-1}(x, v) = (x, g(x)v)$. The map g is called a **transition function**, and we could in fact define vector bundles using such transition functions.

To make vector bundles into the objects of a category, we will now define the morphisms.

Definition 4.1.2. *Let $\pi_1 : E_1 \to B_1$ and $\pi_2 : E_2 \to B_2$ be two vector bundles (not necessarily of the same rank). A **bundle map** is a pair (f, g) of continuous maps $f : B_1 \to B_2$ and $g : E_1 \to E_2$ such that*

1. $f \circ \pi_1 = \pi_2 \circ g$, *that is, the following diagram commutes,*

$$\begin{tikzcd} E_1 \arrow[r, "g"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[r, "f"] & B_2 \end{tikzcd}$$

2. g *restricts to a linear map between fibers* $\pi_1^{-1}(\{x\}) \to \pi_2^{-1}(\{f(x)\})$ *for every* $x \in B_1$.

*An **isomorphism** of vector bundles is a bundle map that has an inverse that is a bundle map as well.*

It is clear from the definition that the identity is a bundle map, that the composition of two bundle maps is again a bundle map, and that composing with $(\mathrm{id}_B, \mathrm{id}_E)$ in either order yields the original bundle map. Thus, vector bundles and bundle maps form a category. We note that since the projection π_1 is surjective, the map f is completely determined by g. Thus, we sometimes denote a bundle map only by the map $g : E_1 \to E_2$ between the total spaces.

For every topological space B, we can consider the subcategory of vector bundles $\pi : E \to B$ over B, where we take the morphisms to be those bundle maps whose map between base spaces is the identity. Regarding such maps, the following lemma will be useful in what follows.

Lemma 4.1.3. *Let $\pi_1 : E_1 \to B$ and $\pi_2 : E_2 \to B$ be vector bundles over the same base space B. Assume that (id_B, f) is a bundle map such that for each point $b \in B$, f maps the fiber $\pi_1^{-1}(b)$ isomorphically onto the corresponding fiber $\pi_2^{-1}(b)$. Then (id_B, f) is an isomorphism of vector bundles.*

Proof. The map f is clearly bijective, so (id_B, f) has the inverse (id_B, f^{-1}) which maps fibers isomorphically. We only need to show that f^{-1} is continuous. It suffices to show that f is continuous on $\pi_2^{-1}(U)$, where $U \subset B$ is an open set over which both E_1 and E_2 trivialize. Consider the sequence of maps

$$U \times \mathbb{R}^n \xleftarrow{\phi} \pi_1^{-1}(U) \xrightarrow{f} \pi_2^{-1}(U) \xrightarrow{\psi} U \times \mathbb{R}^n,$$

where ϕ and ψ are local trivializations. The composition $\psi \circ f \circ \phi^{-1}$ is given by $(b, v) \mapsto (b, g_b v)$, where g_b is an invertible matrix for all $b \in U$, since we assume f to be a fiberwise isomorphism. As we saw in the discussion following Definition 4.1.1, g_b depends continuously on b, and so does the inverse g_b^{-1}. Thus the inverse $(\psi \circ f \circ \phi^{-1})^{-1} = \phi \circ f^{-1} \circ \psi^{-1}$, given by $(b, v) \mapsto (b, g_b^{-1} v)$, is continuous. Since ϕ and ψ are homeomorphisms, f^{-1} must be continuous. $\square$

We introduce some more terminology to highlight a few important classes of bundle maps. If (f, g) is a bundle map, we say that the map $g : E_1 \to E_2$ **covers** $f : B_1 \to B_2$, if it takes each fiber in E_1 with a linear isomorphism onto the corresponding fiber in E_2. An isomorphism between the bundles $\pi : E \to B$ and $B \times \mathbb{R}^n \to B$ is called a **trivialization** of E, and E is called a **trivial bundle**. Besides trivial bundles, other major examples of vector bundles are the tangent bundle TM of a smooth manifold M, and, given an embedding of M into a Euclidean space, the normal bundle of M. We will later define vector bundles of fundamental importance over the Grassmannians $G_n(\mathbb{C}^k)$ and G_n.

Definition 4.1.4. *A **section** of a vector bundle $\pi : E \to B$ is a continuous map $s : B \to E$ satisfying $\pi \circ s = \mathrm{id}_B$.*

According to the definition, a section associates a vector in the fiber $\pi^{-1}(\{x\})$ to each point $x \in B$ in a continuous way. Every vector bundle has a canonical section, the zero section s_0, which maps each point to the zero vector in the corresponding fiber. We will next prove two rather intuitive facts about the zero section, which will nevertheless play a role when we investigate cohomology of vector bundles.

Proposition 4.1.5. *Let $\pi : E \to B$ be a vector bundle of rank n. Firstly, the image of the zero section s_0 is homeomorphic with the base space B. Secondly, the total space E deformation retracts onto this image.*

Proof. Since the zero vector in the fiber $\pi^{-1}(\{x\})$ does not depend on the trivialization, s_0 is well defined. By definition, s_0 is bijective onto its image, and its inverse, the restriction of π, is continuous. Thus we only have to show that s_0 itself is continuous. Since B has a cover of open sets U over which the bundle trivializes, it suffices to consider the restriction of s_0 to such an open set. So assume that $U \subset B$ is open and there exists a homeomorphism $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$. The composition $\phi \circ s_0$ is given by $(\phi \circ s_0)(x) = (x, 0)$ for all x. This composition is clearly continuous, and since ϕ is a homeomorphism, s_0 is also continuous.

To prove the second assertion, define the map $F : E \times I \to E$ so that if (U, ϕ) is a local trivialization of E, then F is locally defined as the composition

$$\pi^{-1}(U) \times I \xrightarrow{\phi \times \mathrm{id}_I} U \times \mathbb{R}^n \times I \xrightarrow{(y,v,t) \mapsto (y,tv)} U \times \mathbb{R}^n \xrightarrow{\phi^{-1}} \pi^{-1}(U).$$

Since F is a composition of continuous maps, it is itself continuous. Furthermore, $F(x, 0) = x$ for all $x \in E$, $F(x, 1)$ equals the zero vector in the fiber $\pi^{-1}(\{x\})$ for all $x \in E$, and $F(x, t) = x$ for all $x \in \mathrm{Im}(s_0)$ and all $t \in I$. To finish the proof that F is a deformation retraction, we need to show that F is well-defined. Let (U, ψ) be another local trivialization, and denote $\psi(x) = (y, w)$. Then the point $(x, t) \in E \times I$ maps to both $\phi^{-1}(y, tv)$ and $\psi^{-1}(y, tw)$. The composition $\psi \circ \phi^{-1}$ is given by

$$(y, v) \ \mapsto \ (y, g(y)v) = (y, w)$$

for some continuous map $g : U \to GL_n(\mathbb{R})$. Hence

$$(\psi \circ \phi^{-1})(y, tv) = (y, g(y)(tv)) = (y, tg(y)v) = (y, tw).$$

Since $\psi \circ \phi^{-1}$ maps (y, tv) to (y, tw), these two points must have the same preimage in E. Thus,

$$\phi^{-1}(y, tv) = \psi^{-1}(y, tw),$$

and F is well-defined. □

It now follows from Corollary 2.2.7 that for every vector bundle $\pi : E \to B$ the projection map π induces isomorphisms of homology and cohomology groups of E and B.

Sections $s_1, ..., s_k$ are called **linearly independent**, if the vectors $s_1(x), ..., s_k(x)$ are linearly independent in the fiber $\pi^{-1}(\{x\})$ for every x. A central question in the study of vector bundles is the existence of linearly independent sections. A first result in this direction is the following.

Proposition 4.1.6. *A vector bundle of rank n is trivial if and only if it possesses n linearly independent sections.*

Proof. Assume first that $\pi : E \to B$ is a trivial vector bundle with trivialization $\phi : E \to B \times \mathbb{R}^n$. Define sections $s_j : B \to E$ by $s_j(b) = \phi^{-1}(b, e_j)$, where e_j is the jth standard coordinate vector. These n sections are evidently continuous, and they are linearly independent since ϕ^{-1} is a fiberwise linear isomorphism.

Now assume that $s_1, ..., s_n$ are linearly independent sections of E. Define the map $\psi : B \times \mathbb{R}^n \to E$ by

$$(b, x_1, ..., x_n) \mapsto x_1 s_1(b) + \cdots + x_n s_n(b).$$

As in the proof of Proposition 4.1.5, we see that ψ is well-defined and continuous, since the definition does not depend on the trivialization, and the composition with any trivialization is continuous. Linear independence of the sections $s_1, ..., s_n$ implies that ψ is a bundle map covering the identity id_B, so by Lemma 4.1.3, ψ is an isomorphism, and E is trivial. $\square$

4.2 Operations on Vector Bundles

4.2.1 Pullback Bundles

Let $\pi : E \to B$ be a vector bundle of rank n, and let $f : B' \to B$ be a continuous map. We define the **pullback bundle** $\pi' : f^*E \to B'$ induced by f as follows. The total space f^*E is the collection of pairs

$$f^*E = \{ (b', e) \in B' \times E \mid f(b') = \pi(e) \}$$

endowed with the subspace topology from the product $B' \times E$. The projection $\pi' : f^*E \to B'$ is defined by $\pi'(b', e) = b'$. It is continuous by definition. The fiber

$$\pi'^{-1}(\{b'\}) = \{ (b', e) \in B' \times E \mid e \in \pi^{-1}(\{f(b')\}) \}$$

is given the same vector space structure as the fiber $\pi^{-1}(\{f(b')\})$. Local trivializations are constructed as follows. Let (U, ϕ) be a local trivialization of E. Define

$$\psi : \pi'^{-1}(f^{-1}(U)) \to f^{-1}(U) \times \mathbb{R}^n$$

by $(b', e) \mapsto (b', v)$, where $\phi(e) = (f(b'), v)$. The map ψ is clearly bijective, and it is continuous since both of its components are continuous. Furthermore, the components of the inverse

$$\psi^{-1} : (b', v) \mapsto (b', \phi^{-1}(f(b'), v))$$

are continuous, so ψ^{-1} is continuous. Thus, ψ is a homeomorphism.

When we classify complex vector bundles using the tautological vector bundle over G_n, we will need the following result.

Lemma 4.2.1. *Let (f, g) be a bundle map of vector bundles $\pi_1 : E_1 \to B_1$ and $\pi_2 : E_2 \to B_2$. If g covers f, then E_1 is isomorphic to the induced bundle f^*B_2.*

Proof. Define the bundle map $(\mathrm{id}_{B_1}, h) : E_1 \to f^*E_2$ by

$$h(e) = (\pi_1(e), g(e)).$$

The map h is continuous by definition, and it maps each fiber of E_1 isomorphically onto the corresponding fiber of f^*B_2. By Lemma 4.1.3, (id_B, h) is an isomorphism. $\square$

If $\pi : E \to B$ is a vector bundle and A is a subspace of B, we call the vector bundle

$$\pi|_{\pi^{-1}(A)} : \pi^{-1}(A) \to A$$

the **restriction** of E to A. It is easy to see that this bundle is isomorphic to the pullback bundle of the inclusion $A \hookrightarrow B$.

4.2.2 Product Bundles

Let $\pi_1 : E_1 \to B_1$ and $\pi_2 : E_2 \to B_2$ be vector bundles of ranks n and m respectively. We define their **product bundle** to be the map

$$\pi_1 \times \pi_2 : E_1 \times E_2 \longrightarrow B_1 \times B_2$$

that takes the point (e_1, e_2) to $(\pi_1(e_1), \pi_2(e_2))$. This map is continuous by definition of the product topology, and the fiber

$$(\pi_1 \times \pi_2)^{-1}(\{(b_1, b_2)\}) = \pi_1^{-1}(\{b_1\}) \times \pi_2^{-1}(\{b_2\})$$

is isomorphic as a vector space to $\mathbb{R}^{n+m}$. If (U_1, ϕ_1) and (U_2, ϕ_2) are local trivializations of E_1 and E_2 respectively, then $(U_1 \times U_2, \phi_1 \times \phi_2)$ is a local trivialization of $E_1 \times E_2$. Thus, $E_1 \times E_2$ has the structure of a vector bundle of rank $n + m$.

4.2.3 Whitney Sums

Using the product and pullback constructions together, we can now define what is perhaps the most important operation on vector bundles. Let $\pi_1 : E_1 \to B$ and $\pi_2 : E_2 \to B$ be vector bundles of ranks n and m, respectively, over the same base space, and let $\Delta : B \to B \times B$ be the diagonal embedding that takes the point b to (b, b). We define the **Whitney sum** of E_1 and E_2 to be the pullback of the product bundle by Δ:

$$\begin{aligned} E_1 \oplus E_2 &= \Delta^*(E_1 \times E_2) \\ &= \{ (b, e_1, e_2) \in B \times E_1 \times E_2 \mid \pi_1(e_1) = \pi_2(e_2) = b \}. \end{aligned}$$

It is a vector bundle of rank $n + m$ over the base space B, and the fiber over $b \in B$ is the direct sum $\pi_1^{-1}(\{b\}) \oplus \pi_2^{-1}(\{b\})$.

As an example from differential geometry, given an embedding $f : M \to \mathbb{R}^k$ of a smooth manifold into a Euclidean space, we can consider the tangent bundle TM and the normal bundle NM of M, given by

$$TM = \{ (p, v) \in M \times \mathbb{R}^k \mid v \in T_pM \}$$

and

$$NM = \{ (p, v) \in M \times \mathbb{R}^k \mid v \perp T_pM \},$$

where T_pM is the tangent space of M at p. Since at each point, the direct sum of the tangent and normal spaces equals the ambient Euclidean space, the Whitney sum $TM \oplus NM$ is actually a trivial bundle. If we take $M = S^2 \subset \mathbb{R}^3$, it follows from the hairy ball theorem that the tangent bundle TS^2 is nontrivial. On the other hand the normal bundle NS^2 is trivial. This example shows that the Whitney sum of a non-trivial bundle with a trivial one may be a trivial bundle. A vector bundle is called **stably trivial** if its Whitney sum with a trivial bundle of some rank is trivial.

Continuing the discussion from the previous section on the relationship between triviality of a vector bundle and existence of sections, we now make a short remark concerning Whitney sums. A **sub-bundle** of a vector bundle $\pi : E' \to B$ is a vector bundle $\pi : E \to B$ such that $E \subset E'$, and each fiber of E is a subspace of the corresponding fiber of E'. We could define something called a **Euclidean metric** on vector bundles, a way of continuously associating an inner product to each fiber of the bundle. Such a Euclidean metric exists for example for any bundle over a paracompact Hausdorff space. Given a Euclidean metric on E', we could define the **orthogonal complement** $E^\perp$ of a sub-bundle E in E'. The following holds.

Remark 4.2.2. *If E is a sub-bundle of a vector bundle E' with a Euclidean metric, then E' splits as the Whitney sum $E' = E \oplus E^\perp$.*

Now, if E has rank n and possesses k linearly independent sections, it can be shown that the sections span a trivial sub-bundle of rank k, and we obtain the following.

Remark 4.2.3. *If E has rank n and possesses k linearly independent sections, then E splits as the Whitney sum $E = T \oplus T^{\perp}$, where T is a trivial bundle of rank k, and $T^{\perp}$ has rank $n-k$.*

See pages 28 and 39 from [13] for proofs of these facts.

4.3 Complex Vector Bundles and Orientability

When studying cohomology of vector bundles, we will concentrate on a special class of vector bundles, namely complex vector bundles. These bundles carry certain additional structure called orientation. We will next define oriented and complex vector bundles, and show that complex vector bundles are indeed oriented.

An **orientation** of a real vector space V is an equivalence class of ordered bases of V, where two bases are considered equivalent if the invertible linear transformation taking one basis to the other has positive determinant. Thus, a real vector space of dimension n has exactly two orientations, corresponding to the two connected components of the general linear group $GL_n(\mathbb{R})$. We call the orientation of $\mathbb{R}^n$ determined by the standard basis $(e_1, ..., e_n)$ the **standard orientation**.

Definition 4.3.1. *An **orientation** on a vector bundle $\pi : E \to B$ is an assignment of orientation in the fiber $\pi^{-1}(\{b\})$ for each $b \in B$ satisfying the following local compatibility condition: for each $b \in B$, there exists a trivialization $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ with $b \in U$ that carries the orientation on each fiber over U to the standard orientation on $\mathbb{R}^n$.*

If E and E' are oriented vector bundles of ranks n and m respectively, then we define the orientation of $E \times E'$ as follows. Let F and F' be fibers of E and E' respectively. If the orientations of F and F' are represented by bases $(v_1, ..., v_n)$ and $(w_1, ..., w_m)$, then the orientation of $F \times F'$ is represented by $(v_1, ..., v_n, w_1, ..., w_m)$. The orientation of $E \oplus E'$ is defined similarly.

Examples of oriented vector bundles include tangent bundles of orientable smooth manifolds, and similarly tangent bundles of nonorientable manifolds are examples of nonorientable vector bundles. Another example of a nonorientable vector bundle can be given by considering the Möbius band as a line bundle over the circle.

An important class of oriented bundles are the complex vector bundles.

Definition 4.3.2. *A **complex vector bundle** of rank n is a continuous map of topological spaces $\pi : E \to B$ such that for each $x \in B$,*

1. *the fiber $\pi^{-1}(\{x\}) \subset E$ has the structure of a complex vector space of dimension n,*
2. *there exists an open neighborhood $U \subset B$ of x and a homeomorphism $\phi : \pi^{-1}(U) \to U \times \mathbb{C}^n$ that restricts to a complex linear isomorphism $\pi^{-1}(\{y\}) \to \{y\} \times \mathbb{C}^n$ for each $y \in U$.*

We define morphisms of complex vector bundles analogously to the real case, making the additional requirement that the fiberwise maps are complex linear.

By forgetting the additional complex linear structure on fibers, we can regard complex vector bundles of rank n as real vector bundles of rank $2n$. It is possible in some cases to reverse this process and give a real vector bundle of even rank the structure of a complex vector bundle as follows. A **complex structure** on a real vector bundle $\pi : E \to B$ of rank $2n$ is a bundle map $J : E \to E$ covering the identity id_B, satisfying $J(J(e)) = -e$ for each $e \in E$, where $-e$ is to be understood as the negative of e in the vector space structure of the corresponding fiber. If such a map exists, we can turn each fiber into a complex vector space by defining complex scalar multiplication by

$$(x + yi)v = xv + J(yv).$$

Local triviality can be checked as follows. For $p \in B$, let $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^{2n}$ be a local trivialization with $b \in U$. By Proposition 4.1.6, there exist $2n$ linearly independent sections on U. At the base point b, these restrict to a real basis for the fiber $\pi^{-1}(\{b\})$ and J restricts to a linear complex structure on the same fiber. Thus, we can choose n of the $2n$ sections, say $s_1, ..., s_n$, such that

$$\{s_1(b), Js_1(b), ..., s_n(b), Js_n(b)\}$$

is also a real basis for the fiber. Then $\{s_1(b), ..., s_n(b)\}$ is a basis over the complex numbers. By continuity, the sections $\{s_1, Js_1, ..., s_n, Js_n\}$ restrict to a real basis at each fiber over a possibly smaller neighborhood U' of b, so the sections $\{s_1, ..., s_n\}$ restrict to a complex basis. Thus, we have found n complex linearly independent sections over U'. By the complex analog of Proposition 4.1.6, there exists a fiberwise complex linear homeomorphism $\psi : \pi^{-1}(U') \to U' \times \mathbb{C}^n$, so the bundle is locally trivial over the neighborhood U'.

We wish to show that the real vector bundle underlying a complex vector bundles has a canonical orientation. To do this, we first declare the complex plane $\mathbb{C}$ to be oriented by the real basis $(1, i)$, corresponding to the standard orientation of $\mathbb{R}^2$.

Proposition 4.3.3. *Every finite dimensional complex vector space has a canonical orientation.*

Proof. If $A \in GL_n(\mathbb{C})$ is the matrix transforming one basis to another, then by Theorem 2.1.6 there is a continuous path of invertible matrices connecting A to the identity matrix I. Embedding $GL_n(\mathbb{C})$ into $GL_{2n}(\mathbb{R})$ in the standard way, the determinant of the corresponding matrices in $GL_{2n}(\mathbb{R})$ cannot change sign along such a continuous path, so the two bases define the same orientation. $\square$

Since each fiber of a complex vector bundle is a complex vector space, all the fibers automatically have the same orientation. Thus, we have the following.

Corollary 4.3.4. *Every complex vector bundle is oriented.*

4.4 Tautological Bundles Over the Grassmannians

We will now define the most central vector bundles appearing in this work. These are the tautological bundles over the complex Grassmannians $G_n(\mathbb{C}^k)$ and G_n.

Definition 4.4.1. *The **tautological bundle** over the finite Grassmannian manifold is the vector bundle*

$$\pi : E^n_k \to G_n(\mathbb{C}^k)$$

with total space

$$E^n_k = \{\, (X, v) \in G_n(\mathbb{C}^k) \times \mathbb{C}^k \mid v \in X \,\},$$

topologized as the subspace of the product $G_n(\mathbb{C}^k) \times \mathbb{C}^k$. The projection map is given by $\pi(X, v) = X$. The fiber over X has the obvious complex linear structure of X. Similarly, the tautological bundle over the infinite Grassmannian is the vector bundle $\pi : E^n \to G_n$ whose total space is

$$E^n = \{\, (X, v) \in G_n \times \mathbb{C}^\infty \mid v \in X \,\}.$$

To be assured that the tautological bundle really is a vector bundle, we need the following result.

Proposition 4.4.2. *The tautological bundle $\pi : E^n_k \to G_n(\mathbb{C}^k)$ is locally trivial.*

Proof. Let $X \in G_n(\mathbb{C}^k)$, and let $U_X = \{Y \in G_n(\mathbb{C}^k) \mid Y \cap X^\perp = 0\}$ be the open neighborhood of X defined in section 1.2. As in section 1.2, we have the projection $p_Y : Y \to X$ and the map $T_Y : X \to X^\perp$. Now define the map $\phi_X : \pi^{-1}(U_X) \to U_X \times X$ by

$$\phi_X(Y, y) = (Y, p_Y(y)).$$

This is continuous and bijective by the definition of U_X. The inverse $\phi_X^{-1} : U_X \times X \to \pi^{-1}(U_X)$ is given by

$$\phi_X^{-1}(Y, x) = (Y, x + T_Y(x)),$$

which is continuous since $Y \mapsto T_Y$ is. Since X can be identified with $\mathbb{C}^n$, the pair (U_X, ϕ_X) is a local trivialization. □

The corresponding result for the bundle E^n requires some care with the direct limit topology.

Proposition 4.4.3. *The tautological bundle* $\pi : E^n \to G_n$ *is locally trivial.*

Proof. As in the previous proof, let $X \in G_n$ be a fixed n-plane in $\mathbb{C}^\infty$. Then X is contained in $\mathbb{C}^N$ for some N. The orthogonal projection $p_X : \mathbb{C}^\infty \to X$ is continuous, since it is continuous when restricted to each $\mathbb{C}^m$ with $m \geq N$. Define U_X to be the set of n-planes $Y \in G_n$ such that p_X restricted to Y is a surjection. Now U_X is open, since the intersection $U_X \cap G_n(\mathbb{C}^k)$ is open for all k, as we have seen earlier.

Define the map $\phi_X : \pi^{-1}(U_X) \to U_X \times X$ by the same formula as in the previous proof. This is continuous since the projection p is. The inverse is also given by the above formula, and we know that it is continuous on each set $(U_X \cap G_n(\mathbb{C}^k)) \times X$. The result now follows from Proposition 2.1.1, which shows continuity the of ϕ^{-1}. □

In the case $n = 1$ we get bundles of rank one over the projective spaces $\mathbb{CP}^k$ and $\mathbb{CP}^\infty$. In each case, the bundle is called the **tautological line bundle**. These bundles play a crucial role in what follows. The next result shows that we have obtained our first examples of non-trivial vector bundles.

Theorem 4.4.4. *The tautological line bundles over* $\mathbb{CP}^k$ *and* $\mathbb{CP}^\infty$ *are non-trivial.*

Proof. By Proposition 4.1.6, a line bundle is trivial if and only if it possesses a non-vanishing section. Consider first the finite case. Assume that $s : \mathbb{CP}^k \to E^1_k$ is a non-vanishing section. The complex projective space $\mathbb{CP}^k$ admits the canonical projection $S^{2k+1} \to \mathbb{CP}^k$ from the unit sphere in $\mathbb{C}^{k+1}$, so composing s with this projection yields a continuous map $S^{2k+1} \to E^1_k$. By the definition of the total space E^1_k, this map is given by $x \mapsto (x, t(x)x)$, where $t : S^{2k+1} \to \mathbb{C}$ is a continuous map satisfying $t(cx) = \frac{1}{c}t(x)$ for all $c \in S^1 \subset \mathbb{C}$. In particular, t is odd, that is, $t(-x) = -t(x)$.

Since s is non-vanishing, t does not assume the value 0, so we can compose with the radial projection $\mathbb{C} \to S^1$. This composition preserves antipodal points, so we obtain an odd map $S^{2k+1} \to S^1$. But by the Borsuk-Ulam theorem (see p. 176 of [6]), every continuous map $S^n \to \mathbb{R}^n$ maps some pair of antipodal points to the same point. An odd map cannot achieve this, so we arrive in a contradiction.

The infinite case follows from the finite case, since for all k, a section $G_n \to E^n$ restricts to a section $G_n(\mathbb{C}^k) \to E^n_k$. □

4.5 Classification of Complex Vector Bundles

In this section we discuss the significance of the complex Grassmannians in the study of complex vector bundles. More precisely, we will show that for every complex n-bundle over a paracompact base space B, there exists a map $B \to G_n$ covered by a bundle map to the tautological bundle. In other words, every complex vector bundle is the pullback of the tautological bundle over some Grassmannian. To construct the required map, we need a lemma.

Lemma 4.5.1. *Assume that* B *is a paracompact Hausdorff space, and let* $\pi : E \to B$ *be a vector bundle. There exists a locally finite open cover* $\{U_k\}_{k=1}^{\infty}$ *of* B *such that* E *trivializes over each set* U_k*.*

Proof. The base space B admits an open cover $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ such that E trivializes over each V_α, and since B is paracompact, we may assume that this collection is locally finite. Since B is paracompact and Hausdorff, by Proposition 2.1.5 there exists an open cover $\{W_\alpha\}_{\alpha \in \mathcal{A}}$ such that $\overline{W}_\alpha \subset V_\alpha$. Since B is normal, by Corollary 2.1.3, there exists a collection $\{\lambda_\alpha\}_{\alpha \in \mathcal{A}}$ of continuous maps $B \to [0, 1]$ such that for all $\alpha \in \mathcal{A}$, λ_α is identically 1 on $\overline{W}_\alpha$ and identically 0 outside V_α.

For each nonempty finite subset S of $\mathcal{A}$, define the set

$$U(S) = \{\, b \in B \mid \min_{\alpha \in S} \lambda_\alpha(b) > \max_{\alpha \notin S} \lambda_\alpha(b) \,\}.$$

We make some observations on the sets $U(S)$. Firstly, since the collection $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ is locally finite, only a finite number of the functions λ_α are nonzero at a given point $b \in B$, so the maximum in the definition of $U(S)$ is well-defined. Secondly, since the function defined by

$$\min_{\alpha \in S} \lambda_\alpha(b) - \max_{\alpha \notin S} \lambda_\alpha(b)$$

is continuous, $U(S)$ is open for all S. Thirdly, assume S and S' have the same number of elements but $S \neq S'$, and fix $\alpha \in S \setminus S'$ and $\beta \in S' \setminus S$. If $b \in U(S)$, then $\lambda_\alpha(b) > \lambda_\beta(b)$, so that $b \notin U(S')$. This shows that for a fixed k, the sets $U(S)$ such that $|S| = k$ are all disjoint. Finally, if $\alpha \in S$, then $\lambda_\alpha(b) > 0$ for all $b \in U(S)$, so $U(S) \in V_\alpha$, and thus E trivializes over each $U(S)$.

Now, for all k, define

$$U_k = \bigcup_{|S|=k} U(S).$$

Since each U_k is a union of open sets, it is itself open. Furthermore, since E trivializes over each $U(S)$ and U_k is a disjoint union of these sets, E trivializes over U_k.

We will next show that the collection $\{U_k\}_{k=1}^{\infty}$ covers B. Let $b \in B$ and let $S \subset \mathcal{A}$ be the set of indices α for which $\lambda_\alpha(b) > 0$. Since $b \in W_\alpha$ for some α, S is nonempty, and since $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ is locally finite, S is finite, say $|S| = k$. By the choice of S, $b \in U(S)$, so $b \in U_k$.

Finally, we will show that $\{U_k\}_{k=1}^{\infty}$ is locally finite. Let $b \in B$. There is an open set U_b containing b that intersects only a finite number of the sets V_α. Let this finite collection be indexed by $S \subset \mathcal{A}$ and let $|S| = k$. If $|S'| > |S|$, then at each point of $U(S')$, more than k of the functions λ_α are nonzero, so $U(S')$ cannot intersect U_b. □

Theorem 4.5.2. *Let* B *be a paracompact Hausdorff space, and let* $\pi : E \to B$ *be a complex* n*-bundle. Then there exists a map* $f : B \to G_n$ *covered by a bundle map* $g : E \to E^n$*.*

Proof. Since B is paracompact and Hausdorff, by Lemma 4.5.1, there exists an open cover $\{U_k\}_{k=1}^{\infty}$ of B such that E trivializes over each set U_k. By Proposition 2.1.5, there exists a cover $\{V_k\}_{k=1}^{\infty}$ such that $\overline{V}_k \subset U_k$ for all k, and similarly there exists a cover $\{W_k\}_{k=1}^{\infty}$ such that $\overline{W}_k \subset V_k$ for all k. By Corollary 2.1.3, there exists a collection $\{\lambda_k\}_{k=1}^{\infty}$ of continuous maps $\lambda_k : B \to [0, 1]$ such that λ_k is identically 1 on $\overline{W}_k$ and identically 0 outside V_k.

Since E trivializes over U_k, for each k, there exists a map $\tilde{h}_k : \pi^{-1}(U_k) \to U_k \times \mathbb{C}^n$. Denote by h_k the composition of $\tilde{h}_k$ with the projection $U_k \times \mathbb{C}^n \to \mathbb{C}^n$. Then h_k restricts to an isomorphism on each fiber. Now for each k, define $h'_k : E \to \mathbb{C}^n$ by

$$h'_k(e) = \begin{cases} 0 & \text{if} \quad \pi(e) \notin V_k \\ \lambda_k(\pi(e))h_k(e) & \text{if} \quad \pi(e) \in U_k \end{cases}$$

for all $e \in E$. Let k be fixed. Firstly, the map h'_k is well-defined, since if $\pi(e) \in (B \setminus V_k) \cap U_k$, then $\lambda_k(\pi(e)) = 0$, so $h'_k(e) = 0$. Secondly, h'_k is continuous on U_k since it is a product of continuous maps, and it is continuous on $B \setminus \overline{V}_k$ as a constant function. Since U_k and $B \setminus \overline{V}_k$ cover B, h'_k is continuous. Thirdly, h'_k is linear on every fiber and maps fibers over W_i isomorphically.

Now define $\hat{g} : E \to \oplus_{k=1}^{\infty} \mathbb{C}^n = \mathbb{C}^\infty$ by

$$\hat{g}(e) = (h'_1(e), h'_2(e), ...).$$

Since $\{V_k\}_{k=1}^{\infty}$ is locally finite, for a fixed $e \in E$, only a finite number of the vectors $h'_k(e)$ are nonzero, so $\hat{g}$ is well-defined. Furthermore, $\hat{g}$ is continuous by the definition of the direct limit topology, and it maps each fiber linearly. Finally, $\hat{g}$ maps each fiber injectively, since each $b \in B$ is contained in some $\overline{W}_k$, and h'_k maps $\pi^{-1}(\{b\})$ injectively.

Define $g : E \to E^n$ by

$$g(e) = (f(e), \hat{g}(e)), \qquad \text{where } f(e) = \hat{g}(\pi^{-1}(\pi(e))).$$

The map g is clearly well-defined and maps each fiber isomorphically. To show that g is continuous, we only need to show that f is. Let $U \subset B$ be an open set over which E trivializes. By Proposition 4.1.6, there exist linearly independent sections $s_1, ..., s_n : U \to E$. We can write g as the composition

$$B \xrightarrow{\hat{f}} \widetilde{V}_n(\mathbb{C}^\infty) \xrightarrow{q} G_n,$$

where $\hat{f}(b) = (\hat{g}(s_1(b)), ..., \hat{g}(s_n(b)))$. Since both $\hat{f}$ and q are continuous, f is continuous. Now (f, g) is the desired bundle map. □

If the base space is compact, we get the following analogous result concerning the finite Grassmannians. The proof is similar but simpler, since this time we obtain a finite open cover $\{U_k\}_{k=1}^{M}$ covering B such that E trivializes over each U_k, and we can define a map $E \to \oplus_{k=1}^{M} \mathbb{C}^n = \mathbb{C}^{nM}$ corresponding to the map $\hat{g}$ above. We omit details of the proof.

Theorem 4.5.3. *Assume that B is a compact Hausdorff space, and let $\pi : E \to B$ be a complex n-bundle. There exists a map $f : B \to G_n(\mathbb{C}^N)$ covered by a bundle map, provided that N is sufficiently large.*

Theorem 4.5.2 shows that the Grassmannian G_n is central in the study of complex vector bundles. However, the relationship between complex vector bundles and the Grassmannian is even stronger. We say that two vector bundles $E_1 \to B$ and $E_2 \to B$ over a common base space are isomorphic if there exists a bundle isomorphism $E_1 \to E_2$ covering the identity map id_B. Let $E_1 \to B$ and $E_2 \to B$ be two complex n-bundles, and let $f_1 : B \to G_n$ and $f_2 : B \to G_n$ be maps covered by bundle maps from E_1 and E_2 to the tautological bundle respectively. It can be shown that E_1 and E_2 are isomorphic if and only if f_1 and f_2 are homotopic.

This can be rephrased in categorical language as follows. Let B be paracompact and Hausdorff, and denote by $\mathcal{E}_n(B)$ the category of complex vector bundles of rank n over B. We can turn $\mathcal{E}_n(-)$ into a contravariant functor from the homotopy category of spaces to sets by sending a continuous map $f : A \to B$ to the pullback operation $f^* : \mathcal{E}_n(B) \to \mathcal{E}_n(A)$. Then, by the statement in the previous paragraph, this functor is represented by G_n, meaning that there is a natural isomorphisms of functors from $\mathcal{E}_n(-)$ to the "functor of points" $[-, G_n]$. This latter functor is defined by sending the space B to the set $[B, G_n]$ of homotopy classes of maps $B \to G_n$, and the homotopy class of $f : A \to B$ to the composition operation

$$[g : B \to G_n] \mapsto [g \circ f : A \to G_n].$$

Thus, the Grassmannian G_n is sometimes called the **classifying space** of complex vector bundles and denoted by $BU(n)$. We merely mention that this notation stems from that fact that G_n is also the classifying space of the unitary group $U(n)$. For more discussion, see [12] or [9].

Chapter 5

Cohomology of Vector Bundles

The study of cohomology of vector bundles is based on the concept of characteristic classes. We will first give a general description of these in an informal fashion. Then we will move on to a detailed discussion of certain specific examples of characteristic classes, the Euler class and Chern classes.

The cohomology rings $H^*(-;R)$ for various commutative rings R are examples of a more general concept of a cohomology theory. In general, these are contravariant functors $k^*(-)$ from some category of spaces to abelian groups satisfying certain axioms. To concentrate on complex vector bundles, consider the functor $\mathcal{E}_n(-)$ and a given cohomology theory $k^*(-)$. Recall from section 4.5 that $\mathcal{E}_n(-)$ sends a space B to the set of equivalence classes of complex n-bundles over B, and, modulo homotopy, a map $f : A \to B$ to the precomposition operation $g \mapsto g \circ f$. A **characteristic class** c is a natural transformation from $\mathcal{E}_n(-)$ to $k^*(-)$. In other words, for a given isomorphism class of vector bundles $E \to B$, it associates a cohomology class $c(E) \in k^*(B)$. In addition, this association is natural, meaning that if $f : A \to B$ is covered by a bundle map, then $c(\mathcal{E}_n(f)E) = k^*(f)c(E)$, on more concisely $c(f^*E) = f^*c(E)$.

Since the functor $\mathcal{E}_n(-)$ is represented by the infinite Grassmannian G_n, it follows from the Yoneda lemma of category theory (see [11]) that the set of characteristic classes corresponding to a given cohomology theory $k^*(-)$ are in bijection with the cohomology classes in $k^*(G_n)$. Concretely, we obtain the characteristic classes of a given bundle $E \to B$ by pulling back along a map $f : B \to G_n$ that is covered by a bundle map. For this reason, a central task in the theory of characteristic classes is to compute the cohomology groups $k^*(G_n)$. We will achieve this in the case of singular cohomology with integer coefficients in Theorem 5.3.9. For more discussion on characteristic classes, see [12].

Similarly, characteristic classes can be defined to other classes of vector bundles by considering an appropriate functor in place of $\mathcal{E}_n$ and an appropriate classifying space in place of G_n.

5.1 Thom Isomorphism

To fix some notation, if V is a vector space, we will denote by V_0 the punctured space $V \setminus \{0\}$. Similarly, if E is the total space of a vector bundle, E_0 will denote the space obtained by removing the zero section from E.

In Section 4.3 we defined an orientation of a real vector space V as a choice of equivalence class of ordered bases of V. We will now give an equivalent formulation in terms of cohomology. If V has dimension n, then the groups $H_n(V, V_0)$ and $H^n(V, V_0; \mathbb{Z})$ are infinite cyclic groups. We define the orientation of V to be a choice of generator, called the **orientation class**, for either of these groups. The correspondence between the different formulations is the following. Given an ordered basis of V, let $\sigma : \Delta^n \to V$ be a singular n-simplex embedded linearly into V such that an interior point of Δ^n is mapped to the origin, and that the basis formed by the images of the vectors $v_i - v_0$ along the edges of Δ^n gives the preferred orientation of V. The homology class of σ will be denoted by μ_V, and it is a generator of $H_n(V, V_0)$. The corresponding generator of $H^n(V, V_0; \mathbb{Z})$ is denoted by u_V and is represented by a cocycle ϕ such that $\phi(\sigma) = 1$. Thus, according to the isomorphism $H^n(V, V_0; \mathbb{Z}) \cong \mathrm{Hom}(H_n(V, V_0), \mathbb{Z})$, we have $u_V(\mu_V) = 1$.

Let now $\pi : E \to B$ be a real vector bundle of rank n. An orientation in each fiber F over b determines a generator $u_b \in H^n(F, F_0; \mathbb{Z})$ for each $b \in B$, and vice versa. This generator is called the **orientation class** of the fiber. The next result shows how the local compatibility condition is related to the formulation of orientation in terms of cohomology.

Lemma 5.1.1. *Assume that the orientation of fibers in the* n*-bundle* $\pi : E \to B$ *satisfies the local compatibility condition in Definition 4.3.1. Then* B *can be covered with neighborhoods* U *such that there exists a cohomology class*

$$u \in H^n(\pi^{-1}(U), \pi^{-1}(U)_0; \mathbb{Z})$$

that for each $b \in U$ *restricts to the preferred generator* $u_b \in H^n(\pi^{-1}(\{b\}), \pi^{-1}(\{b\})_0; \mathbb{Z})$ *under the homomorphism* i^* *induced by the inclusion* $i : (\pi^{-1}(\{b\}), \pi^{-1}(\{b\})_0) \hookrightarrow (\pi^{-1}(U), \pi^{-1}(U)_0)$.

Proof. Let $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ be a local trivialization that takes the orientation of each fiber to the standard orientation of $\mathbb{R}^n$. Denote by $u_{\mathbb{R}^n}$ the cohomology class in $H^n(\mathbb{R}^n, \mathbb{R}^n_0; \mathbb{Z})$ that gives this standard orientation. Following the chain of maps

$$H^0(U; \mathbb{Z}) \times H^n(\mathbb{R}^n, \mathbb{R}^n_0; \mathbb{Z}) \xrightarrow{\times} H^n(U \times \mathbb{R}^n, U \times \mathbb{R}^n_0; \mathbb{Z}) \xrightarrow{\phi^*} H^n(\pi^{-1}(U), \pi^{-1}(U)_0; \mathbb{Z}),$$

where the first map is the cross product, let $u = \phi^*(\mathbb{1}_U \times u_{\mathbb{R}^n})$. Let $b \in U$ and let $\sigma : \Delta^n \to \pi^{-1}(\{b\})$ be a singular n-simplex that represents the given orientation class μ_b in $H_n(\pi^{-1}(\{b\}), \pi^{-1}(\{b\})_0)$. Then by the definition of ϕ, the singular n-simplex $p \circ \phi \circ i \circ \sigma : \Delta^n \to \mathbb{R}^n$ represents the standard orientation class $\mu_{\mathbb{R}^n} \in H_n(\mathbb{R}^n, \mathbb{R}^n_0)$, where p is the projection $U \times \mathbb{R}^n \to \mathbb{R}^n$. Thus,

$$i^* u(\mu_b) = u(i_* \mu_b) = 1 \cdot u_{\mathbb{R}^n}(p_* \phi_* i_* \mu_b) = u_{\mathbb{R}^n}(\mu_{\mathbb{R}^n}) = 1,$$

so u satisfies the condition in the statement of the lemma. □

Let now R be any unital commutative ring and let $\Phi : \mathbb{Z} \to R$ be the unique ring homomorphism. The induced homomorphism $\Phi^* : H^n(F, F_0; \mathbb{Z}) \to H^n(F, F_0; R)$ sends the preferred generator u_b to a generator of $H^n(F, F_0; R)$, and for an oriented bundle, the local compatibility condition still holds for a class $u \in H^n(\pi^{-1}(U), \pi^{-1}(U)_0; R)$. This defines an R**-orientation** of the bundle $\pi : E \to B$. Our next goal is to generalize and considerably strengthen Lemma 5.1.1. More precisely, we aim to prove to following result.

Theorem 5.1.2. *Let* $\pi : E \to B$ *be an oriented vector bundle. Then for any coefficient ring* R *there exists a unique cohomology class* $u \in H^n(E, E_0; R)$ *that restricts to give the local* R*-orientation at each fiber. The map* $x \mapsto x \smile u$ *defines an isomorphism* $H^k(E; R) \to H^{k+n}(E, E_0; R)$ *for all* k.

This is called the **Thom isomorphism theorem**. A cohomology class $u \in H^n(E, E_0; R)$ satisfying the property that it restricts to the orientation class at each fiber is called a **fundamental class** of the bundle. In other words, the Thom isomorphism theorem states that each oriented vector bundle admits a unique fundamental class. We note that for zero-dimensional vector bundles the theorem is trivially true, since we can choose $u = \mathbb{1}$.

The proof of Theorem 5.1.2 will be split in several parts. First, we sharpen Lemma 5.1.1 so that the theorem holds for trivial bundles. Next, we extend the theorem to hold for bundles over compact bases B. Then we will prove the theorem for arbitrary base spaces when the coefficient ring R is a field. Finally, we extend the proof to all base spaces and all rings. Although in our later discussion on cohomology of complex vector bundles we will only use the coefficient ring $\mathbb{Z}$, we will prove the Thom isomorphism for more general rings R as this requires no further effort.

For the moment, we will mostly omit the ring R in the notation of cohomology groups. Let us begin with a lemma.

Lemma 5.1.3. *For any ring* R*, there exists an element* $e^n \in H^n(\mathbb{R}^n, \mathbb{R}^n_0; R)$ *such that for any space* B *and any open set* $A \subset B$ *the map* $H^k(B, A; R) \to H^{k+n}(B \times \mathbb{R}^n, B \times \mathbb{R}^n_0 \cup A \times \mathbb{R}^n; R)$ *defined by* $a \mapsto a \times e^n$ *is an isomorphism for all* k*.*

Proof. We will first construct the element e and prove the lemma in the case $n = 1$ and $A = \emptyset$. Denote by $\mathbb{R}_0$ the nonzero real numbers and by $\mathbb{R}_+$ and $\mathbb{R}_-$ the positive and the negative real numbers, respectively. Since the pair $(\mathbb{R}, \mathbb{R}_-)$ deformation retracts onto the pair (p, p), where p is any negative real number, we see that $H^k(\mathbb{R}, \mathbb{R}_-) = 0$ for all k. The long exact sequence of the triple $(\mathbb{R}, \mathbb{R}_0, \mathbb{R}_-)$ begins

$$0 \to H^0(\mathbb{R}, \mathbb{R}_0) \to H^0(\mathbb{R}, \mathbb{R}_-) \to H^0(\mathbb{R}_0, \mathbb{R}_-) \to H^1(\mathbb{R}, \mathbb{R}_0) \to H^1(\mathbb{R}, \mathbb{R}_-) \to \cdots.$$

We thus have an isomorphism $H^0(\mathbb{R}_0, \mathbb{R}_-) \xrightarrow{\delta} H^1(\mathbb{R}, \mathbb{R}_0)$. By excision, the homomorphism

$$H^0(\mathbb{R}_0, \mathbb{R}_-) \xrightarrow{i_0^*} H^0(\mathbb{R}_+)$$

induced by the inclusion $(\mathbb{R}_+, \emptyset) \overset{i_0}{\hookrightarrow} (\mathbb{R}_0, \mathbb{R}_-)$ is an isomorphism. We thus have a sequence of isomorphisms

$$H^0(\mathbb{R}_+) \underset{\cong}{\overset{i_0^*}{\longleftarrow}} H^0(\mathbb{R}_0, \mathbb{R}_-) \underset{\cong}{\overset{\delta}{\longrightarrow}} H^1(\mathbb{R}, \mathbb{R}_0).$$

Denote by $e = e^1$ the image of $\mathbb{1} \in H^0(\mathbb{R}_+)$ under these isomorphisms. Then e is a generator of the free R-module $H^1(\mathbb{R}, \mathbb{R}_0)$. Now consider the diagram

$$\begin{tikzcd}
 & H^0(\mathbb{R}_+) \arrow[d, "a\times"] & H^0(\mathbb{R}_0, \mathbb{R}_-) \arrow[l, "i_0^*"'] \arrow[r, "\delta"] \arrow[d, "a\times"] & H^1(\mathbb{R}, \mathbb{R}_0) \arrow[d, "a\times"] \\
H^k(B) \arrow[r, "\cong"] & H^k(B \times \mathbb{R}_+) & H^k(B \times \mathbb{R}_0, B \times \mathbb{R}_-) \arrow[l, "i^*"'] \arrow[r, "\delta'"] & H^{k+1}(B \times \mathbb{R}, B \times \mathbb{R}_0).
\end{tikzcd}$$

The homomorphism i^* is an isomorphism by excision, and δ' is an isomorphism since the long exact sequence of the triple $(B \times \mathbb{R}, B \times \mathbb{R}_0, B \times \mathbb{R}_-)$ contains the segment

$$H^k(B \times \mathbb{R}, B \times \mathbb{R}_-) \to H^k(B \times \mathbb{R}_0, B \times \mathbb{R}_-) \xrightarrow{\delta'} H^{k+1}(B \times \mathbb{R}, B \times \mathbb{R}_0) \to H^{k+1}(B \times \mathbb{R}, B \times \mathbb{R}_-),$$

and $H^k(B \times \mathbb{R}, B \times \mathbb{R}_-) = H^{k+1}(B \times \mathbb{R}, B \times \mathbb{R}_-) = 0$ since $B \times \mathbb{R}$ deformation retracts onto $B \times \mathbb{R}_-$. The bottom leftmost isomorphism comes from the fact that $B \times \mathbb{R}_+$ deformation retracts onto B. The left square commutes since the two diagrams

$$\begin{tikzcd}
(\mathbb{R}_+, \emptyset) \arrow[r, "i_0"] & (\mathbb{R}_0, \mathbb{R}_-) \\
(B \times \mathbb{R}_0, B \times \mathbb{R}_-) \arrow[u, "pr_{\mathbb{R}_+}"] \arrow[r, "i"'] & (B \times \mathbb{R}_0, B \times \mathbb{R}_-) \arrow[u, "pr_{\mathbb{R}_0}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
B \times \mathbb{R}_- \arrow[r, "i"] \arrow[dr, "pr_B"'] & B \times \mathbb{R}_0 \arrow[d, "p_B"] \\
 & B
\end{tikzcd}$$

commute, and the right square commutes by naturality of the cup product and the long exact sequence of a triple. Following the element $\mathbb{1} \in H^0(\mathbb{R}_+)$ around the diagram, we see that the bottom row of the diagram defines the isomorphism $a \mapsto a \times e$.

Let now A be nonempty, and let $z \in C^1(\mathbb{R}, \mathbb{R}_0)$ represent the generator e. For all k, the rows of the diagram

$$\begin{tikzcd}
0 \arrow[r] & C^k(B,A) \arrow[r, "j^\sharp"] \arrow[d, "\times z"] & C^k(B) \arrow[r, "i^\sharp"] \arrow[d, "\times z"] & C^k(A) \arrow[r] \arrow[d, "\times z"] & 0 \\
0 \arrow[r] & C^{k+1}(B\times\mathbb{R}, B\times\mathbb{R}_0 + A\times\mathbb{R}) \arrow[r, "j^\sharp"] & C^{k+1}(B\times\mathbb{R}, B\times\mathbb{R}_0) \arrow[r, "i^\sharp"] & C^{k+1}(A\times\mathbb{R}, A\times\mathbb{R}_0) \arrow[r] & 0
\end{tikzcd}$$

are by definition exact, and a straightforward calculation shows that both squares commute. In addition, all the maps commute with the coboundary maps δ. Thus, cross product with z induces a chain map of the corresponding long exact sequences in cohomology:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
H^{k-1}(B) \arrow[r, "\times e"] \arrow[d, "i^*"] & H^k(B\times\mathbb{R}, B\times\mathbb{R}_0) \arrow[d, "i^*"] \\
H^{k-1}(A) \arrow[r, "\times e"] \arrow[d, "\delta"] & H^k(A\times\mathbb{R}, A\times\mathbb{R}_0) \arrow[d, "\delta"] \\
H^k(B,A) \arrow[r, "\times e"] \arrow[d, "j^*"] & H^{k+1}(B\times\mathbb{R}, B\times\mathbb{R}_0\cup A\times\mathbb{R}) \arrow[d, "j^*"] \\
H^k(B) \arrow[r, "\times e"] \arrow[d, "i^*"] & H^{k+1}(B\times\mathbb{R}, B\times\mathbb{R}_0) \arrow[d, "i^*"] \\
H^k(A) \arrow[r, "\times e"] \arrow[d] & H^{k+1}(A\times\mathbb{R}, A\times\mathbb{R}_0) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

We know that the two top maps and the two bottom maps are isomorphisms, so it follows from the Five-Lemma that the middle map is an isomorphism as well. This concludes the proof for $n = 1$.

For arbitrary $n \geq 1$, consider first the case $B = \mathbb{R}^{n-1}$, $A = \mathbb{R}_0^{n-1}$. Since $\mathbb{R}_0^n = \mathbb{R}^{n-1}\times\mathbb{R}_0 \cup \mathbb{R}_0^{n-1}\times\mathbb{R}$, it follows from the case $n = 1$ that the map

$$H^{n-1}(\mathbb{R}^{n-1}, \mathbb{R}_0^{n-1}) \xrightarrow{\times e} H^n(\mathbb{R}^n, \mathbb{R}_0^n)$$

is an isomorphism. Inductively define $e^n = e^{n-1}\times e$. To conclude the proof for a general pair (B, A), we see that since the cross product is associative, it follows by induction that the map

$$H^k(B,A) \xrightarrow{\times e^n} H^{k+n}(B\times\mathbb{R}^n, B\times\mathbb{R}_0^n \cup A\times\mathbb{R}^n)$$

is an isomorphism. □

Using this lemma, we will now prove the Thom isomorphism theorem for trivial bundles.

Proposition 5.1.4. *The Thom isomorphism theorem holds for trivial oriented vector bundles.*

Proof. Let $\pi : E \to B$ be a trivial oriented vector bundle of rank n, and let $\phi : E \to B \times \mathbb{R}^n$ be a trivialization. Then ϕ maps the orientation of each fiber of E to either the standard orientation of $\mathbb{R}^n$ or the opposite one. This map is locally constant, so it is constant on connected components. Thus, by composing with a reflection in appropriate connected components, we may assume that ϕ carries the orientation of each fiber to the standard orientation of $\mathbb{R}^n$. In fact, we could simply state that a trivial vector bundle always has a canonical orientation according to the standard orientation of $\mathbb{R}^n$.

By Lemma 5.1.3, we have the isomorphism

$$H^0(B) \xrightarrow{\times e^n} H^n(B \times \mathbb{R}^n, B \times \mathbb{R}^n_0) \xrightarrow{\phi^*} H^n(E, E_0).$$

It is not difficult to see that $e^n \in H^n(\mathbb{R}^n, \mathbb{R}^n_0)$ corresponds to the standard orientation of $\mathbb{R}^n$. Let $b \in B$, $F = \pi^{-1}(\{b\})$, and let $i : F \hookrightarrow E$ be the inclusion. Let $\tilde{\mu}_b : \Delta^n \to F$ represent the element $\mu_b \in H_n(F, F_0)$ corresponding to the orientation of F, and let $\tilde{e}^n \in C^n(\mathbb{R}^n, \mathbb{R}^n_0)$ represent e^n. Then for any

$$x \in H^0(B) = \ker \delta \subset C^0(B),$$

we have

$$\begin{aligned}(i^\sharp \phi^\sharp (x \times \tilde{e}^n))(\tilde{\mu}_b) &= (x \times \tilde{e}^n)(\phi \circ i \circ \tilde{\mu}_b) \\ &= x(pr_B \circ \phi \circ i \circ \tilde{\mu}_b[v_0])\tilde{e}^n(pr_{\mathbb{R}^n} \circ \phi \circ i \circ \tilde{\mu}_b) \\ &= x(b),\end{aligned}$$

since

$$pr_{\mathbb{R}^n} \circ \phi \circ i \circ \tilde{\mu}_b : \Delta^n \to \mathbb{R}^n$$

corresponds to the standard orientation. We have to find a cocycle x such that

$$(i^\sharp \phi^\sharp (x \times \tilde{e}^n))(\tilde{\mu}_b) = x(b) = 1$$

for all $b \in B$, since then $\phi^*(x \times e^n)$ will be the cohomology class u we are looking for. The unique element that has this property is of course $\mathbb{1}_B \in C^0(B)$, that assigns the value 1 to every point of B. Thus, we define $u = \phi^*(\mathbb{1}_B \times e^n)$.

To prove that cup product with u gives an isomorphism $H^k(E) \to H^{k+n}(E, E_0)$, we note that since $B \times \mathbb{R}^n$ deformation retracts onto B, every cohomology class in $x \in H^k(E)$ can be written uniquely as $\phi^*(y \times \mathbb{1}_{\mathbb{R}^n})$ for some $y \in H^k(B)$. Now, using associativity of the cup product, we have

$$\begin{aligned}x \smile u &= \phi^*(y \times \mathbb{1}_{\mathbb{R}^n}) \smile \phi^*(\mathbb{1}_B \times e^n) = \phi^*((y \times \mathbb{1}_{\mathbb{R}^n}) \smile (\mathbb{1}_B \times e^n)) \\ &= \phi^*(pr_B^*(y) \smile pr_{\mathbb{R}^n}^*(\mathbb{1}_{\mathbb{R}^n}) \smile pr_B^*(\mathbb{1}_B) \smile pr_{\mathbb{R}^n}^*(e^n)) \\ &= \phi^*(pr_B^*(y) \smile \mathbb{1}_{B\times\mathbb{R}^n} \smile \mathbb{1}_{B\times\mathbb{R}^n} \smile pr_{\mathbb{R}^n}^*(e^n)) \\ &= \phi^*(pr_B^*(y) \smile pr_{\mathbb{R}^n}^*(e^n)) = \phi^*(y \times e^n).\end{aligned}$$

By Lemma 5.1.3, the association $y \mapsto \phi^*(y \times e^n)$ is an isomorphism. □

Proposition 5.1.5. *The Thom isomorphism theorem holds for oriented vector bundles over compact base spaces.*

Proof. By compactness, the base space can be covered by a finite number of open sets over which the bundle trivializes. Proving the result by induction on the number of open sets in this cover, the initial step is given by Proposition 5.1.4. The induction step reduces to showing that if the base space of the vector bundle $\pi : E \to B$ can be covered by open sets B^1 and B^2 such that the result holds when restricted to B^1, to B^2, and to their intersection $B^3 = B^1 \cap B^2$, then it holds also for B. Denote by E^i and E^i_0 the sets

$\pi^{-1}(B^i)$ and $\pi^{-1}(B^i)_0$, respectively for $i = 1,2,3$. We have the following segment of the Mayer-Vietoris sequence for $E = E^1 \cup E^2$.

$$H^{n-1}(E^3, E_0^3) \to H^n(E, E_0) \xrightarrow{\phi} H^n(E^1, E_0^1) \oplus H^n(E^2, E_0^2) \xrightarrow{\psi} H^n(E^3, E_0^3)$$

Since the Thom isomorphism holds for E^1 and E^2, there exist elements

$$u^1 \in H^n(E^1, E_0^1) \quad \text{and} \quad u^2 \in H^n(E^2, E_0^2)$$

that restrict to the given orientation in each fiber. Furthermore, both elements restrict to the corresponding unique element $u^3 \in H^n(E^3, E_0^3)$. Thus, $(u^1, -u^2)$ maps to $u^3 - u^3 = 0$ under ψ, so by exactness, there is an element $u \in H^n(E, E_0)$ that maps to $(u^1, -u^2)$ under ϕ. Then u restricts to the preferred orientation on each fiber over B. Finally, the Thom isomorphism

$$H^{n-1}(E^3, E_0^3) \cong H^{-1}(E^3) = 0$$

implies that ϕ is an injection, and hence u is uniquely determined.

Now consider the commutative diagram

$$\begin{array}{ccc}
H^{k-1}(E^1) \oplus H^{k-1}(E^2) & \xrightarrow{(\times u^1, \times u^2)} & H^{n+k-1}(E^1, E_0^1) \oplus H^{n+k-1}(E^2, E_0^2) \\
\downarrow & & \downarrow \\
H^{k-1}(E^3) & \xrightarrow{\times u^3} & H^{n+k-1}(E^3, E_0^3) \\
\downarrow & & \downarrow \\
H^k(E) & \xrightarrow{\times u} & H^{n+k}(E, E_0) \\
\downarrow & & \downarrow \\
H^k(E^1) \oplus H^k(E^2) & \xrightarrow{(\times u^1, \times u^2)} & H^{n+k}(E^1, E_0^1) \oplus H^{n+k}(E^2, E_0^2) \\
\downarrow & & \downarrow \\
H^k(E^3) & \xrightarrow{\times u^3} & H^{n+k}(E^3, E_0^3)
\end{array}$$

The columns are segments of Mayer-Vietoris sequences, hence exact. By the Thom isomorphisms corresponding to E^1, E^2 and E^3, the two top and the two bottom horizontal maps are isomorphisms, so by the Five-Lemma 2.2.4, also the middle map is an isomorphism. □

The next step in the proof will be extending the isomorphism theorem for all base spaces B and all coefficient fields.

Proposition 5.1.6. *The Thom isomorphism theorem holds for all oriented vector bundles when the coefficient ring is a field.*

Proof. Let $\pi : E \to B$ be an oriented vector bundle of rank n and let Λ be a field. We will assume that all cohomology groups will have coefficients in Λ and hence omit the coefficient ring from the notation. Since for all spaces X and all k, the group $H^{k-1}(X)$ is a free Λ-module, it follows from the universal coefficient theorem that $H^k(X) \cong \mathrm{Hom}(H_k(X, \Lambda))$, and similarly for relative groups. Since every compact subset of E

is contained in $\pi^{-1}(K)$ for some compact set $K \subset B$, we can use Lemma 2.2.5 and Proposition 2.2.9 to deduce that for all k,

$$\begin{aligned} H^k(E) &\cong \mathrm{Hom}(H_k(E), \Lambda) \cong \mathrm{Hom}(\varinjlim H_k(\pi^{-1}(K_i)), \Lambda) \\ &\cong \varprojlim \mathrm{Hom}(H_k(\pi^{-1}(K_i)), \Lambda) \\ &\cong \varprojlim H^k(\pi^{-1}(K_i)), \end{aligned}$$

where the limits are taken over the directed set of compact subspaces of X. Here the assumption of field coefficients allows us to shift from cohomology to homology, after which we can use the fact that homology is compactly supported. Similarly,

$$H^k(E, E_0) \cong \varprojlim H^k(\pi^{-1}(K_i), \pi^{-1}(K_i)_0).$$

By Proposition 5.1.5, for each i, there exists a unique fundamental class $u_i \in H^n(\pi^{-1}(K_i), \pi^{-1}(K_i)_0)$ that restricts to the orientation class of each fiber. Thus, the element $(u_i)_{i \in I}$ maps to an element $u \in H^n(E, E_0)$ that has the same property, and this element is unique since the map is an isomorphism.

To show that the map $H^k(E) \xrightarrow{\smile u} H^{k+n}(E, E_0)$ is an isomorphism, for each i, consider the commutative diagram

$$\begin{tikzcd} H^k(E) \arrow[r, "\smile u"] \arrow[d] & H^{k+n}(E, E_0) \arrow[d] \\ H^k(\pi^{-1}(K_i)) \arrow[r, "\smile u_i"] & H^{k+n}(\pi^{-1}(K_i), \pi^{-1}(K_i)_0), \end{tikzcd}$$

where the vertical maps are induced by the inclusion $K_i \hookrightarrow E$. The bottom map is an isomorphism by Proposition 5.1.5. As we pass to the inverse limit in the lower row, we obtain the diagram

$$\begin{tikzcd} H^k(E) \arrow[r, "\smile u"] \arrow[d] & H^{k+n}(E, E_0) \arrow[d] \\ \varprojlim H^k(\pi^{-1}(K_i)) \arrow[r, "\smile (u_i)_{i \in I}"] & \varprojlim H^{k+n}(\pi^{-1}(K_i), \pi^{-1}(K_i)_0), \end{tikzcd}$$

where also the vertical maps have become isomorphisms. Thus, the top vertical map is an isomorphism. □

The final step in the proof of Theorem 5.1.2 is to extend the result for all rings R. To do this, we need a lemma.

Lemma 5.1.7. *Assume that there exists a fundamental class $u \in H^n(E, E_0; \mathbb{Z})$ for the rank n vector bundle $\pi : E \to B$. For any ring R, let $u_R \in H^n(E, E_0; R)$ be the image of u under the map $H^n(E, E_0; \mathbb{Z}) \to H^n(E, E_0; R)$ induced by the unique ring homomorphism $\mathbb{Z} \to R$. Then the maps $H_{n+k}(E, E_0; R) \to H_n(E; R)$, given by the cap product $\sigma \mapsto u_R \frown \sigma$, and the map $H^k(E; R) \to H^{n+k}(E, E_0; R)$, given by the cup product $\phi \mapsto u_R \smile \phi$, are isomorphisms for all k.*

Proof. Let $v \in C^n(E, E_0; \mathbb{Z})$ be a cochain representing the fundamental class u, and denote by v_R its image in $C^n(E, E_0; R)$. Since v is a cocycle, it follows that for any $\sigma \in C_k(E, E_0)$,

$$\partial(\sigma \frown v) = (\partial \sigma) \frown v,$$

so the map $C_k(E, E_0) \to C_{k-n}(E)$ given by $\sigma \mapsto \sigma \frown \nu$ defines a chain map $C_*(E, E_0) \to C_*(E)$ of degree $-n$. For any ring R, the induced map $C^{k-n}(E; R) \to C^k(E, E_0; R)$ is given by $\phi \mapsto \phi \smile \nu_R$. Thus, the induced map $H^{k-n}(E; R) \to C^k(E, E_0; R)$ is given by $\phi \mapsto \phi \smile u_R$. If R is a field, then we know by Proposition 5.1.6 that this induced map is an isomorphism. Using Proposition 2.2.3, we now conclude that the induced map is an isomorphism for all homology and cohomology groups with arbitrary coefficients. □

We are now finally ready to prove the Thom isomorphism theorem in the general case.

Proof of Theorem 5.1.2. Let $\pi : E \to B$ be an oriented vector bundle of rank n. For any compact subset $K \subset B$, denote by $u_K \in H^n(\pi^{-1}(K), \pi^{-1}(K)_0; \mathbb{Z})$ the fundamental class of the restriction of E to K. This class exists and is unique by Proposition 5.1.5. By Lemma 5.1.7, the map

$$H_{n-1}(\pi^{-1}(K), \pi^{-1}(K)_0) \xrightarrow{\frown u_K} H_{-1}(\pi^{-1}(K)) = 0$$

is an isomorphism. Using now the isomorphism

$$H_{n-1}(E, E_0) \cong \varinjlim H_{n-1}(\pi^{-1}(K_i), \pi^{-1}(K_i)_0),$$

where the inverse limit is taken over all compact subsets of B, we conclude that the homology group $H_{n-1}(E, E_0)$ is zero. It now follows from the Universal Coefficient Theorem that

$$H^n(E, E_0; \mathbb{Z}) \cong \mathrm{Hom}(H_n(E, E_0), \mathbb{Z}),$$

so just as in the proof of Proposition 5.1.5, we have

$$H^n(E, E_0; \mathbb{Z}) \cong \varprojlim H^n(\pi^{-1}(K_i), \pi^{-1}(K_i)_0; \mathbb{Z}).$$

Since each group on the right hand side of the equation has a unique fundamental class u_i, it follows that the element $(u_i)_{i\in I}$ maps to the unique element $u \in H^n(E, E_0; \mathbb{Z})$ that restricts to the orientation class in each fiber. Now, for any ring, using the homomorphism $H^n(E, E_0; \mathbb{Z}) \to H^n(E, E_0; R)$ induced by $\mathbb{Z} \to R$, we obtain a fundamental class $u_R \in H^n(E, E_0; R)$, although we do not know if this class is unique. Nevertheless, we can use Lemma 5.1.7 to conclude that the map $H^k(E; R) \to H^{k+n}(E, E_0; R)$ given by $\phi \mapsto u \smile \phi$ is an isomorphism for all k. It remains to show that u_R is unique for every ring R. But the map

$$H^0(E; R) \xrightarrow{\smile u} H^n(E, E_0; R)$$

is an isomorphism, so every fundamental class $u' \in H^n(E, E_0; R)$ must be of the form $\phi \smile u$ for some $\phi \in H^0(E; R)$, and since the elements of $H^0(E; R)$ are represented by locally constant maps $E \to R$, the only choice that works is $\phi = \mathbb{1}$. □

Now that we have established the existence and uniqueness of the fundamental class for any oriented vector bundle, it is straightforward to check some of its basic properties.

Proposition 5.1.8. *Let $\pi : E \to B$ and $\pi' : E' \to B'$ be oriented vector bundles of rank n and m, respectively. Let their fundamental classes be $u \in H^n(E, E_0; R)$ and $u' \in H^n(E', E'_0; R)$.*

1. *If the map $f : B \to B'$ is covered by a bundle map $E \to E'$ that maps each fiber of E with an orientation preserving linear isomorphism to the corresponding fiber of E', then $u = f^*u'$.*

2. *If the orientation of E is changed, then u changes sign.*

3. *The fundamental class of $E \times E'$ is $u \times u'$.*

Proof. Let $b \in B$ and $F = \pi^{-1}(b)$. Let $i : F \hookrightarrow E$ be the inclusion. If the generator $\mu_b \in H_n(F, F_0)$ represents the orientation of F, then $i^*f^*u'(\mu_b) = u'(f_*i_*\mu_b) = 1$, since f preserves the orientation and u' is the fundamental class of E'. Since this holds for every fiber of E, it follows from the uniqueness of the fundamental class that $u = f^*u'$. This proves the first part. Similarly, since changing the sign of u changes the sign of the induced class in each fiber, the second part follows from the uniqueness of u.

For the third part, we note first that $E_0 \subset E$ and $E'_0 \subset E'$ are open subsets and that

$$E \times E'_0 \cup E_0 \times E' = (E \times E')_0,$$

so the cross product

$$H^n(E, E_0; R) \times H^m(E', E'_0; R) \to H^{n+m}(E \times E', (E \times E')_0; R)$$

is defined. By uniqueness of the fundamental class, we must show that $u \times u'$ restricts to the orientation class at each fiber. Let $b'' = (b, b') \in B \times B'$ be a basepoint, let

$$i : F'' = (\pi \times \pi')^{-1}(b'') \to E \times E'$$

be the inclusion, and let $\mu \in H_n(F'', F''_0)$ represent the orientation. Let

$$pr : E \times E' \to E \quad \text{and} \quad pr' : E \times E' \to E'$$

be the projections. Then

$$i^*(u \times u')(\mu) = (u \times u')(i_*\mu) = u(pr_*i_*\mu)u'(pr'_*i_*\mu) = 1,$$

since u and u' are fundamental classes and $pr_*i_*\mu$ and $pr'_*i_*\mu$ represent the orientations of the fibers $\pi^{-1}(b)$ and $\pi^{-1}(b')$. It follows from the uniqueness of the fundamental class that $u \times u'$ is the fundamental class of $E \times E'$.

□

5.2 Euler Class

We are now ready to define our first characteristic class associated to vector bundles. From now on, we will use the integers as the coefficient ring of cohomology groups. Let $\pi : E \to B$ be an oriented vector bundle of rank n. The inclusion $i : (E, \emptyset) \hookrightarrow (E, E_0)$ induces a homomorphism

$$i^* : H^n(E, E_0) \to H^n(E).$$

On the other hand, as we have seen earlier, since E deformation retracts onto B, the induced homomorphism $\pi^* : H^n(B) \to H^n(E)$ is an isomorphism.

Definition 5.2.1. *The* ***Euler class*** $e(E)$ *of the oriented* n*-bundle* $\pi : E \to B$ *is the image in* $H^n(B)$ *of the fundamental class* $u \in H^n(E, E_0)$ *under the sequence of maps*

$$H^n(E, E_0) \xrightarrow{i^*} H^n(E) \xrightarrow{\pi^{*-1}} H^n(B).$$

In other words, it is the unique element of $H^n(B)$ *that satisfies the equation*

$$\pi^* e(E) = i^* u.$$

The next proposition describes some of the most basic properties of the Euler class. In particular, the first part shows that the Euler class is indeed a characteristic class.

Proposition 5.2.2. *The Euler class satisfies the following properties.*

1. *The Euler class is natural with respect to bundle maps. More precisely, if* $f : B \to B'$ *is a continuous map covered by a bundle map* $E \to E'$, *then* $e(E) = f^*e(E')$.
2. *If the orientation of the bundle is reversed, then the Euler class changes sign.*
3. *If the vector bundle has odd rank, then* $2e(E) = 0$.

Proof. To prove the first part, we note that commutativity of the diagram

$$\begin{tikzcd} (E, E_0) \arrow[d, "g"] & (E, \varnothing) \arrow[l, "i"'] \arrow[r, "\pi"] \arrow[d, "g"] & B \arrow[d, "f"] \\ (E', E'_0) & (E', \varnothing) \arrow[l, "i'"'] \arrow[r, "\pi'"] & B' \end{tikzcd}$$

induces commutativity of the diagram

$$\begin{tikzcd} H^n(E, E_0) \arrow[r, "i^*"] & H^n(E) & H^n(B) \arrow[l, "\pi*"'] \\ H^n(E', E'_0) \arrow[u, "g^*"] \arrow[r, "i'^*"] & H^n(E') \arrow[u, "g^*"] & H^n(B'). \arrow[l, "\pi'^*"'] \arrow[u, "f^*"] \end{tikzcd}$$

By the first part of Proposition 5.1.8, g^* maps the fundamental class $u' \in H^n(E', E'_0)$ to the fundamental class $u \in H^n(E, E_0)$, so following the diagram around proves that $f^*e(E') = e(E)$.

The second part follows immediately from the second part of Proposition 5.1.8. For the third part we note that if the rank of the bundle is odd, then the continuous map $g : E \to E$ taking a point $v \in E$ to its negative inside the fiber is an orientation reversing bundle map taking each fiber isomorphically onto itself. Thus, on one hand, the Euler class changes its sign. On the other hand, the map from B to itself induced by g is the identity, so it maps the Euler class to itself. The statement then follows from the equation $e(E) = -e(E)$. □

Characteristic classes are designed to measure the extent to which a vector bundle deviates from being a trivial bundle. The next result shows that the Euler class provides one such measure.

Proposition 5.2.3. *If the oriented bundle* $\pi : E \to B$ *possesses a nonzero section, then the Euler class* $e(E)$ *vanishes.*

Proof. Let $s : B \to E_0$ be a nonzero section. Then the composition

$$B \xrightarrow{s} E_0 \xhookrightarrow{i} E \xrightarrow{\pi} B$$

is the identity on B, and hence the induced composition

$$H^n(B) \xrightarrow{\pi^*} H^n(E) \xrightarrow{i^*} H^n(E_0) \xrightarrow{s^*} H^n(B)$$

is the identity on $H^n(B)$. By the definition of the Euler class, $\pi^*e(E) = j^*u$, where $j^* : H^n(E, E_0) \to H^n(E)$ is the canonical map. But the sequence

$$H^n(E, E_0) \xrightarrow{j^*} H^n(E) \xrightarrow{i^*} H^n(E_0)$$

is part of the long exact sequence of the pair (E, E_0), hence the composition i^*j^* is zero. Thus,

$$e(E) = s^*i^*\pi^*e(E) = s^*i^*j^*u = s^*0 = 0.$$

□

Proposition 5.2.4. *Let* $\pi : E \to B$ *and* $\pi' : E' \to B'$ *be oriented vector bundles of rank* n *and* m*, respectively.*

1. *The Euler classes satisfy the identity* $e(E \times E') = e(E) \times e(E')$.
2. *Assume* $B = B'$. *Then* $e(E \oplus E') = e(E) \smile e(E')$.

Proof. By the third part of Proposition 5.1.8, the fundamental class of $E \times E'$ is $u \times u'$, where u and u' are the fundamental classes of E and E' respectively. Let $i : (E, \varnothing) \to (E, E_0)$ and $i' : (E', \varnothing) \to (E', E_0')$ be the inclusions, and let $i_1 : (E \times E', \varnothing) \to (E \times E', E_0 \times E')$ and $i_2 : (E \times E', \varnothing) \to (E \times E', E \times E_0')$ be the induced inclusions of the products. Furthermore, let $\tilde{i} : (E \times E', \varnothing) \to (E \times E', (E \times E')_0)$ be the inclusion. The diagrams

$$\begin{tikzcd} H^n(E, E_0) \arrow[r, "i^*"] \arrow[d, "pr_E^*"] & H^n(E) \arrow[d, "pr_E^*"'] \\ H^n(E \times E', (E \times E')_0) \arrow[r, "i_2^*"] & H^n(E \times E') \end{tikzcd} \qquad \begin{tikzcd} H^n(B) \arrow[r, "\pi^*"] \arrow[d, "pr_B^*"'] & H^n(E) \arrow[d, "pr_E^*"] \\ H^n(B \times B') \arrow[r, "(\pi \times \pi')^*"] & H^n(E \times E') \end{tikzcd}$$

and the corresponding diagrams for E' commute. Thus, by naturality of the cup product,

$$\begin{aligned} (\pi \times \pi')^* e(E \times E') &= \tilde{i}^*(u \times u') = \tilde{i}^*(pr_E^* u \smile {pr'_E}^* u') \\ &= i_1^* pr_E^* u \smile i_2^* pr_{E'}^* u' \\ &= pr_E^* i^* u \smile pr_{E'}^* i'^* u' \\ &= pr_E^* \pi^* e(E) \smile pr_{E'}^* \pi'^* e(E') \\ &= (\pi \times \pi')^* pr_B^* e(E) \smile (\pi \times \pi')^* pr_{B'}^* e(E') \\ &= (\pi \times \pi')^* (pr_B^* e(E) \smile pr_{B'}^* e(E')) \\ &= (\pi \times \pi')^* (e(E) \times e(E')). \end{aligned}$$

Thus, by the definition of the Euler class, $e(E \times E') = e(E) \times e(E')$.

To prove the second assertion, we consider the diagonal embedding $\Delta : B \to B \times B$. On one hand, Δ is by definition covered by a bundle map $E \oplus E' \to E \times E'$ that takes each fiber isomorphically onto the corresponding fiber. Thus, by the first part of Proposition 5.2.2, $\Delta^* e(E \times E') = e(E \oplus E')$. On the other hand, for either factor, the composition $B \xrightarrow{\Delta} B \times B \xrightarrow{pr_B} B$ is clearly the identity on B. Thus,

$$\begin{aligned} \Delta^* e(E \times E') &= \Delta^*(e(E) \times e(E')) \\ &= \Delta^*(pr_B^* e(E) \smile pr_B^* e(E')) \\ &= \Delta^* pr_B^* e(E) \smile \Delta^* pr_B^* e(E') \\ &= e(E) \smile e(E'). \end{aligned}$$

□

We conclude the discussion of cohomology of oriented vector bundles with a variant of the long exact sequence associated to the pair (E, E_0).

Proposition 5.2.5. *Let* $\pi : E \to B$ *be an oriented vector bundle of rank* n*, and let* e *be its Euler class. Let* π_0 *denote the composition* $E_0 \hookrightarrow E \xrightarrow{\pi} B$. *Then the following **Gysin sequence** is exact.*

$$\cdots \to H^k(B) \xrightarrow{\smile e} H^{k+n}(B) \xrightarrow{\pi_0^*} H^{k+n}(E_0) \to H^{k+1}(B) \to \cdots$$

Proof. We begin with the long exact sequence of the pair (E, E_0)

$$\cdots \to H^{k+n}(E, E_0) \xrightarrow{j^*} H^{k+n}(E) \xrightarrow{i^*} H^{k+n}(E_0) \xrightarrow{\delta} H^{k+n+1}(E, E_0) \to \cdots .$$

By the Thom isomorphism $H^k(E) \xrightarrow{\smile u} H^{k+n}(E, E_0)$, we get the sequence

$$\cdots \to H^k(E) \xrightarrow{g^*} H^{k+n}(E) \to H^{k+n}(E_0) \to H^{k+1}(E) \to \cdots ,$$

where $g^*(x) = j^*(x \smile u)$. By properties of the relative cup product, we have $j^*(x \smile u) = x \smile j^*(u)$. Now we use the isomorphism $\pi^* : H^*(B) \to H^*(E)$ to replace the cohomology groups of E with cohomology groups of B:

$$\cdots \to H^k(B) \to H^{k+n}(B) \to H^{k+n}(E_0) \to H^{k+1}(B) \to \cdots$$

The map $H^{k+n}(B) \to H^{k+n}(E_0)$ is now $i^*\pi^* = \pi_0^*$, and the map $H^k(B) \to H^{k+n}(B)$ in the sequence is given by $(\pi^*)^{-1} g^* \pi^*$. But

$$\begin{aligned}
(\pi^*)^{-1} g^* \pi^*(x) &= (\pi^*)^{-1}(\pi^*(x) \smile j^*(u)) \\
&= (\pi^*)^{-1}(\pi^*(x) \smile \pi^*(e)) \\
&= (\pi^*)^{-1}\pi^*(x) \smile (\pi^*)^{-1}\pi^*(e) \\
&= x \smile e.
\end{aligned}$$

□

Corollary 5.2.6. *For $k < n - 1$, the map $H^k(B) \xrightarrow{\pi_0^*} H^k(E_0)$ is an isomorphism.*

Proof. This follows from the Gysin sequence, since the groups $H^{k-n}(B)$ and $H^{k-n+1}(B)$ are zero by definition. □

5.3 Chern Classes and the Cohomology Ring of the Grassmannian

5.3.1 Definition of Chern Classes

We will now define Chern classes, which are characteristic classes for complex vector bundles. As we saw earlier, complex vector bundles have a canonical orientation, so in particular, the Euler class of the underlying real vector bundle is defined. We will define Chern classes in terms of Euler classes. To achieve this, we first construct an auxiliary $(n-1)$-bundle for every complex n-bundle. Then by repeatedly performing this construction, we define Chern classes as pullbacks of the Euler classes of the various auxiliary bundles. The idea of the construction is the following. If the original bundle is $\pi : E \to B$, then the base space of the new bundle will be the punctured total space E_0. Since a point $e \in E_0$ is a nonzero vector in the fiber over $\pi(e)$, we could define the fiber in the new bundle over this point to be the orthogonal complement of e in the fiber over $\pi(e)$. Unfortunately we do not necessarily have a notion of inner product defined consistently in the whole bundle E. To avoid this problem, we define the fiber over e to be the quotient space of the fiber over $\pi(e)$ by the one-dimensional subspace spanned by e.

Let $\pi : E \to B$ be a complex vector bundle of rank n, and let E_0 denote the complement of the zero section in E, as usual. First, we define the set

$$\hat{E} = \{ (e, v) \in E_0 \times E \mid v \in \pi^{-1}(\pi(e)) \}$$

consisting of pairs (e, v) of a nonzero vector $e \in E_0$ and a vector v belonging to the same fiber as e. We give $\hat{E}$ the subspace topology. Next, we define an equivalence relation $\sim$ on $\hat{E}$ so that $(e, v_1) \sim (e, v_2)$ if and only if $v_1 - v_2$ is a scalar multiple of e. Thus, each equivalence class can be written as the quotient vector space $\pi^{-1}(\pi(e))/\langle e\rangle$, where $\langle e\rangle$ is the one-dimensional subspace spanned by e. Let $\tilde{E}$ be the quotient space $\hat{E}/\sim$ endowed with the quotient topology, and let $q : \hat{E} \to \tilde{E}$ be the canonical map. We have a projection map $\tilde{\pi} : \tilde{E} \to E_0$ given by $\tilde{\pi}([(e,v)]) = e$. This is well-defined and continuous, since the composition $\tilde{\pi} \circ q$, given by $(e,v) \mapsto e$, is continuous. To show that $\tilde{\pi} : \tilde{E} \to E_0$ is a complex vector bundle, we have to show local triviality.

Lemma 5.3.1. *Consider $E = \mathbb{C}^n$ as a complex n-bundle over a point. Then $\tilde{\pi} : \tilde{E} \to E_0$ is locally trivial.*

Proof. We have $\hat{E} = \mathbb{C}^n_0 \times \mathbb{C}^n$. In $\tilde{E}$, two points (e,v_1) and (e,v_2) of $\hat{E}$ are identified if and only if

$$v_1 - v_2 = ce \quad \text{for some } c \in \mathbb{C}.$$

For $1 \le i \le n$, set

$$U_i^+ = \{(z_1,...,z_n) \in \mathbb{C}^n \mid \operatorname{Re} z_i > 0\} \subset \mathbb{C}^n_0$$

and

$$U_i^- = \{(z_1,...,z_n) \in \mathbb{C}^n \mid \operatorname{Re} z_i < 0\} \subset \mathbb{C}^n_0.$$

Clearly the sets $U_i^\pm$ are open and cover $\mathbb{C}^n_0$. Let now U be either U_i^+ or U_i^-, and define

$$\hat{\phi} : U \times \mathbb{C}^n \to U \times \mathbb{C}^{n-1}$$

as follows. Write a point $(e,v) \in U \times \mathbb{C}^n$ as

$$(e,v) = (e, ke + (z_1,...,z_n)),$$

where $k \in \mathbb{C}$ and $e \cdot (z_1,...,z_n) = 0$. Here k and $(z_1,...,z_n)$ are uniquely determined, since $e \neq 0$. We define $\hat{\phi}$ by

$$(e, ke + (z_1,...,z_n)) \mapsto (e, (z_1,...,z_{i-1}, z_{i+1},...,z_n)).$$

If (e,v_1) and (e,v_2) belong to the same equivalence class in $\tilde{E}$, then v_1 and v_2 differ by a scalar multiple of e, so $\hat{\phi}$ sends them to the same point in $U \times \mathbb{C}^{n-1}$. Thus, we can define the function

$$\phi : \tilde{E} \to U \times \mathbb{C}^{n-1}, \quad [(e,v)] \mapsto \hat{\phi}(e,v).$$

Now ϕ is continuous since $\hat{\phi}$ is, and a continuous inverse of ϕ is given by sending $(e,(z,...,z_{n-1}))$ to the class of $(e,(z_1,...,z_{i-1},a,z_{i+1},...,z_{n-1}))$, where $e \cdot (z_1,...,z_{i-1},a,z_{i+1},...,z_{n-1}) = 0$. Here a is uniquely determined since $e_i \neq 0$. Hence, ϕ is a homeomorphism, and since it clearly preserves fibers, we have shown that $\tilde{E}$ is locally trivial. □

Proposition 5.3.2. *For any complex n-bundle $\pi : E \to B$, the bundle $\tilde{\pi} : \tilde{E} \to E_0$ is locally trivial.*

Proof. It suffices to prove the proposition for trivial bundles, since any bundle can be covered by patches of trivial ones. For a space B, let $E = B \times \mathbb{C}^n$, so that $\hat{E} = B \times \mathbb{C}^n_0 \times \mathbb{C}^n$. Let U_i and $\hat{\phi}$ be as in the lemma. Then $\mathrm{id}_B \times \hat{\phi} : B \times U_i \times \mathbb{C}^n \to B \times U_i \times \mathbb{C}^{n-1}$ induces a bundle isomorphism $\tilde{E} \to B \times U_i \times \mathbb{C}^{n-1}$, and since $B \times U_i$ is open in $B \times \mathbb{C}^n_0$, we are done. □

We are now in a position to define Chern classes. Let $\pi : E \to B$ be a complex vector bundle of rank n. By the **Euler class** $e(E)$ of the complex vector bundle we mean the Euler class of the underlying real bundle. Recall that by Corollary 5.2.6, the map $H^k(B) \xrightarrow{\pi_0^*} H^k(E_0)$ is an isomorphism whenever $k < 2n - 1$.

Definition 5.3.3. *The **Chern classes** $c_i(E) \in H^{2i}(B)$ are defined as follows. The **top Chern class** $c_n(E)$ is equal to the Euler class $e(E)$. For $i < n$, we define by induction the Chern class $c_i(E)$ to be the unique element in $H^{2i}(E)$ satisfying the equation*

$$\pi_0^* c_i(E) = c_i(\tilde{E}).$$

*For $i > n$, the Chern classes are defined to be zero. The **total Chern class** is the sum*

$$c(E) = \sum_{i \in \mathbb{Z}} c_i(E) = 1 + c_1(E) + ... + c_n(E) \in H^*(B).$$

We will now prove some of the most basic properties of Chern classes. In particular, we will show that they are characteristic classes.

Proposition 5.3.4. *Let $\pi : E \to B$ and $\pi' : E' \to B'$ be complex n-bundles, and assume that $f : B \to B'$ is covered by a bundle map $E \to E'$ .Then $c_i(E) = f^* c_i(E')$ for all i.*

Proof. We argue by induction on n. If $n = 0$, the only nonzero Chern classes are $c_0(E)$ and $c_0(E')$, and

$$f^* c_0(E') = f^* \mathbb{1}_{B'} = \mathbb{1}_B = c_0(E).$$

Assume now that the result holds for bundles of rank at most $n - 1$. Since the top Chern class $c_n(E)$ is the Euler class, it follows from the first part of Proposition 5.2.2 that

$$c_n(E) = e(E) = f^* e(E') = f^* c_n(E').$$

Let now $i < n$. Commutativity of the diagram

$$\begin{tikzcd} E_0 \arrow[r, "g"] \arrow[d, "\pi_0"'] & E_0' \arrow[d, "\pi_0'"] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$

implies that $g^* \pi_0'^* = \pi_0^* f^*$. Furthermore, we have the commutative diagram

$$\begin{tikzcd} \tilde{E} \arrow[r, "\tilde{g}"] \arrow[d, "\tilde{\pi}_0"'] & \tilde{E}' \arrow[d, "\tilde{\pi}_0'"] \\ E_0 \arrow[r, "g"'] & E_0' \end{tikzcd}$$

where $\tilde{g}$ takes the equivalence class of (e, v) to the equivalence class of $(g(e), g(v))$. Clearly, $\tilde{g}$ is well-defined, continuous, and covers g. Since $\tilde{E}$ and $\tilde{E}'$ are complex bundles of rank $n - 1$, it follows from the induction hypothesis that $c_i(\tilde{E}) = g^* c_i(\tilde{E})'$. Now, using the definition of Chern classes, we have

$$\pi_0^* f^* c_i(E') = g^* \pi_0'^* c_i(E') = g^* c_i(\tilde{E}') = c_i(\tilde{E}),$$

so $f^* c_i(E') = c_i(E)$. ☐

A characteristic class is called **stable** if its value on a vector bundle remains invariant under taking a a Whitney sum of the bundle with a trivial bundle. The Euler class is clearly nonstable, since taking a Whitney sum increases the rank of the bundle. In contrast, the next proposition shows that Chern classes are stable.

Proposition 5.3.5. *Let* $\pi : E \to B$ *be a complex* n*-bundle. If* $T \to B$ *is a trivial bundle, then* $c(E \oplus T) = c(E)$.

Proof. We may identify T with $B \times \mathbb{C}^m$ for some m, and since $B \times \mathbb{C}^m \cong (B \times \mathbb{C}^{m-1}) \times \mathbb{C}$, the general case follows from the case $m = 1$ by induction. Let $E' = E \oplus (B \times \mathbb{C})$ and denote the projection $E' \to B$ by π'. We can describe E' as the set of tuples $(b, e, z) \in B \times E \times \mathbb{C}$ such that $\pi(e) = b$. The bundle E' possesses the obvious nonzero section $s : B \to E'_0$ given by $s(b) = (b, 0, 1)$. Furthermore, this section is covered by the bundle map $\tilde{s} : E \to \tilde{E}'$ which sends the point e to the equivalence class of $(\pi(e), 0, 1, e, 0)$ in $\tilde{E}'$, which in our case is a quotient of a subset of the space $B \times E \times \mathbb{C} \times E \times \mathbb{C}$.

Firstly, since E' possesses a nonzero section, we have by Proposition 5.2.3 that $c_{n+1}(E') = 0$, and since E is an n-bundle, also $c_{n+1}(E) = 0$. Secondly, for $i \leq n$, the existence of $\tilde{s}$ implies that $s^* c_i(\tilde{E}') = c_i(E)$, and by the definition of Chern classes we have $c_i(\tilde{E}') = \pi_0'^* c_i(E')$. Since $\pi'_0 \circ s$ is the identity map of B, the map $s^* \pi_0'^*$ is the identity on $H^*(B)$. Thus,

$$c_i(E) = s^* c_i(\tilde{E}') = s^* \pi_0'^* c_i(E') = c_i(E').$$

□

5.3.2 Cohomology of the Projective Space

We will now compute the cohomology ring of the complex projective space $\mathbb{CP}^k$ using Chern classes. Let $\pi : L_k \to \mathbb{CP}^k$ be the tautological line bundle, where $L_k = E^1_{k+1}$ as defined in Section 4.4. Recall that the total space L_k is the set of pairs (X, v), where X is a line through the origin in $\mathbb{C}^{k+1}$ and v is a vector in X.

Theorem 5.3.6. *The cohomology ring* $H^*(\mathbb{CP}^k; \mathbb{Z})$ *of the complex projective space is the truncated polynomial ring* $\mathbb{Z}[c_1(L_k)]/(c_1(L_k)^{k+1})$ *generated by the first Chern class of the tautological line bundle* L_k *and terminating in dimension* $2k$.

Proof. Denote $E = L_k$ and $c = c_1(L_k)$. The complement of the zero section, E_0, may be identified with the punctured complex vector space $\mathbb{C}^{k+1} \setminus \{0\}$, an explicit homeomorphism given by $(X, v) \mapsto v$. The space $\mathbb{C}^{k+1} \setminus \{0\}$ deformation retracts onto the sphere $S^{2k+1} = \{z \in \mathbb{C}^{k+1} \mid |z| = 1\}$, so E_0 has the homotopy type of the sphere S^{2k+1}. Now consider the Gysin sequence

$$\cdots \to H^{i+1}(E_0) \to H^i(\mathbb{CP}^k) \xrightarrow{\smile c} H^{i+2}(\mathbb{CP}^k) \to H^{i+2}(E_0) \to \cdots .$$

Since $H^i(E_0) \cong H^i(S^{2k+1})$ for all i, we have $H^i(E_0) = 0$ for $1 \leq i \leq 2k$, so the Gysin sequence breaks up into segments

$$0 \to H^i(\mathbb{CP}^k) \xrightarrow{\smile c} H^{i+2}(\mathbb{CP}^k) \to 0$$

for $0 \leq i \leq 2k - 2$. This implies that

$$H^0(\mathbb{CP}^k) \cong H^2(\mathbb{CP}^k) \cong \cdots \cong H^{2k}(\mathbb{CP}^k),$$

and

$$H^1(\mathbb{CP}^k) \cong H^3(\mathbb{CP}^k) \cong \cdots \cong H^{2k-1}(\mathbb{CP}^k).$$

By Theorem 3.1.3, $\mathbb{CP}^k$ is path-connected, so $H^0(\mathbb{CP}^k) \cong \mathbb{Z}$, which implies that $H^{2i}(\mathbb{CP}^k) \cong \mathbb{Z}$ for $0 \leq i \leq k$. Furthermore, the Gysin sequence contains the segment

$$H^{-1}(\mathbb{CP}^k) \to H^1(\mathbb{CP}^k) \to H^1(E_0),$$

and since $H^{-1}(\mathbb{CP}^k) = H^1(E_0) = 0$, we have $H^1(\mathbb{CP}^k) = 0$, implying $H^{2i+1}(\mathbb{CP}^k) = 0$ for $0 \leq i \leq k-1$. Finally, since the CW-structure of $\mathbb{CP}^k$ contains no cells of dimension higher than $2k$, the group $H^i(\mathbb{CP}^k)$ vanishes for $i > 2k$ by cellular cohomology. Thus, the cohomology groups have the expected structure, and the isomorphisms $H^{2i}(\mathbb{CP}^k) \xrightarrow{c} H^{2i+2}(\mathbb{CP}^k)$ give the desired ring structure. □

Let now L denote the tautological line bundle over $\mathbb{CP}^\infty$.

Corollary 5.3.7. $H^*(\mathbb{CP}^\infty) \cong \mathbb{Z}[c_1(L)]$.

Proof. Since every compact subspace of $\mathbb{CP}^\infty$ is contained in some $\mathbb{CP}^k \subset \mathbb{CP}^\infty$, we have by Proposition 2.2.9

$$\begin{aligned} H^*(\mathbb{CP}^\infty) &\cong \lim_{k\to\infty} H^*(\mathbb{CP}^k) \\ &\cong \lim_{k\to\infty} \mathbb{Z}[c_1(L_k)]/(c_1(L_k)^{k+1}) \\ &\cong \mathbb{Z}[c_1(L)] \end{aligned}$$

$\square$

5.3.3 Cohomology of the Grassmannian

In this section we will prove the culminating result of this work, namely, we will describe the integral cohomology ring of the infinite Grassmannian G_n. Let us begin with a lemma.

Lemma 5.3.8. *For $n \geq 2$, there exists a map $f : E^n_0 \to G_{n-1}$ such that the induced homomorphism*

$$f^* : H^*(G_{n-1}) \to H^*(E^n_0)$$

is an isomorphism. Furthermore, the composition $\lambda = f^{*-1}\pi_0^* : H^*(G_n) \to H^*(G_{n-1})$ maps the Chern class $c_k(E^n)$ to $c_k(E^{n-1})$ for each k.

Proof. The map f is constructed as follows. A point $(X, v) \in E^n_0$ consists of a plane X in $\mathbb{C}^\infty$ and a nonzero vector v in that plane, so we define $f(X, v)$ to be $X \cap v^\perp$, the orthogonal complement of v inside X with respect to the Hermitian inner product. Since $v \neq 0$ is contained in X, the plane $X \cap v^\perp$ is a well-defined $(n-1)$-dimensional plane, that is, a point in G_{n-1}.

To show that f induces isomorphism on cohomology, let us consider the finite Grassmannian $G_n(\mathbb{C}^N)$ inside G_n for some large $N > n$. Let $E^n_N = \pi^{-1}(G_n(\mathbb{C}^N)) \subset E^n$ be the tautological bundle over $G_n(\mathbb{C}^N)$, and let $f_N : E^n_{N,0} \to G_{n-1}(\mathbb{C}^N)$ be the restriction of f to $E^n_{N,0}$. We can identify f_N with a projection of a certain vector bundle as follows.

Define the set $D = \{ (X, v) \in G_{n-1}(\mathbb{C}^N) \times \mathbb{C}^N \mid v \perp X \}$, endowed with the subspace topology of the product topology. The projection map $\rho : D \to G_{n-1}(\mathbb{C}^N)$ then clearly defines a vector bundle of rank $N - n + 1$. Local triviality can be checked similarly as with the tautological bundle over the Grassmannian. Now, define a map $E^n_{N,0} \to D$ by sending (X, v) to the point $(X \cap v^\perp, v)$. This is clearly continuous, and has a continuous inverse $(Y, v) \mapsto (\langle Y, v \rangle, v)$, where $\langle Y, v \rangle$ denotes the subspace spanned by Y and v. Furthermore, this map takes the fiber

$$f_N^{-1}(Y) = \{ (X, v) \in E^n_{N,0} \mid v \perp Y, X = \langle Y, v \rangle \}$$

onto the fiber of D over Y with a vector space isomorphism. Furthermore, the composition

$$E^n_{N,0} \to D \to G_{n-1}(\mathbb{C}^N)$$

is precisely f_N.

By these remarks, the cohomology groups $H^k(E^n_{N,0})$ and $H^k(D_0)$ are isomorphic for all k. In addition, by Corollary 5.2.6, ρ induces isomorphisms $H^k(G_{n-1}(\mathbb{C}^N)) \cong H^k(D_0)$ for $k \leq 2(N - n)$. Thus, f_N induces isomorphisms $H^k(G_{n-1}(\mathbb{C}^N)) \cong H^k(E^n_{N,0})$ for $k \leq 2(N - n)$. Passing now to the direct limit $N \to \infty$, we see that f induces isomorphism for all cohomology groups $H^k(G_{n-1}) \cong H^k(E^n_0)$.

To show that $\lambda = f^{*-1}\pi_0^*$ maps the Chern class $c_k(E^n)$ to $c_k(E^{n-1})$, consider first the case $k = n$. The top Chern class $c_n(E^n)$ is equal to the Euler class e, and by definition e satisfies the equation $\pi^* e = j^* u$, where u is the fundamental class and j^* is the canonical homomorphism $H^n(E^n, E_0^n) \to H^n(E^n)$. Since $\pi_0^* = i^*\pi^*$, where i^* is induced by the inclusion $E_0^n \to E^n$, we have

$$\pi_0^* c_n(E^n) = i^*\pi^* c_n(E^n) = i^* j^* u = 0,$$

since $i^* j^*$ appears in the long exact sequence of the pair (E^n, E_0^n), and hence is zero. Thus,

$$\lambda c_n(E^n) = 0 = c_n(E^{n-1}),$$

since E^{n-1} is an $(n-1)$-bundle.

Assume now that $k < n$. The map $f : E_0^n \to G_{n-1}$ can be covered by a bundle map $\tilde{f} : \tilde{E}_0^n \to E^{n-1}$ as follows. A point in $\tilde{E}_0^n$ is determined by a plane X, a vector $v \in X$, and an equivalence class $[w]$ of vectors in X, such that $[w] = [w']$ if and only if $w - w' = zv$ for some $z \in \mathbb{C}$. Define $\tilde{f}$ so that it takes the triplet $(X, v, [w])$ to the point $(X \cap v^\perp, w_0)$, where w_0 is the unique vector in the equivalence class $[w]$ orthogonal to v. By Proposition 5.3.4, we now have $c_k(\tilde{E}^n) = f^* c_k(E^{n-1})$. But by the definition of Chern classes we have $c_k(\tilde{E}^n) = \pi_0^* c_k(E^n)$, so that

$$\lambda c_k(E^n) = f^{*-1}\pi_0^* c_k(E^n) = f^{*-1} c_k(\tilde{E}^n) = c_k(E^{n-1}).$$

□

Theorem 5.3.9. *The cohomology ring $H^*(G_n)$ is isomorphic to $\mathbb{Z}[c_1(E^n), ..., c_n(E^n)]$, the polynomial ring over $\mathbb{Z}$ freely generated by the Chern classes of the tautological bundle over G_n.*

Proof. We argue by double induction. Since by Corollary 5.3.7 we know that the result holds for $n = 1$, our main induction hypothesis is that the result holds for $n - 1$ when $n \geq 2$.

Consider the Gysin sequence

$$\cdots \to H^k(G_n) \xrightarrow{\smile c_n(E^n)} H^{k+2n}(G_n) \to H^{k+2n}(E_0^n) \to H^{k+1}(G_n) \to \cdots .$$

By Lemma 5.3.8, we can replace $H^*(E_0^n)$ by $H^*(G_{n-1})$, so we obtain the exact sequence

$$\cdots \to H^k(G_n) \xrightarrow{\smile c_n(E^n)} H^{k+2n}(G_n) \xrightarrow{\lambda} H^{k+2n}(G_{n-1}) \to H^{k+1}(G_n) \to \cdots .$$

By induction, $H^*(G_{n-1})$ is isomorphic to the polynomial ring over $\mathbb{Z}$ generated freely by

$$c_1(E^{n-1}), ..., c_{n-1}(E^{n-1}).$$

The cohomology ring $H^*(G_n)$ contains all polynomial expressions in the Chern classes $c_1(E^n), ..., c_n(E^n)$, and since by the same lemma, $\lambda c_k(E^n) = c_k(E^{n-1})$ for all k, we see that λ is surjective. Thus, the Gysin sequence breaks up into short exact sequences

$$0 \to H^k(G_n) \xrightarrow{\smile c_n(E^n)} H^{k+2n}(G_n) \xrightarrow{\lambda} H^{k+2n}(G_{n-1}) \to 0.$$

Assume first that $k < 0$, so that $H^k(G_n) = 0$, and the map λ is an isomorphism. Let $x \in H^{k+2n}(G_n)$. By the main induction hypothesis, $\lambda(x) = h(c_1(E^{n-1}), ..., c_{n-1}(E^{n-1}))$ for some unique polynomial h. Thus, $x = h(c_1(E^n), ..., c_{n-1}(E^n))$. This shows that every cohomology class of sufficiently low dimension can be expressed as a unique polynomial in the Chern classes. Thus, as our secondary induction hypothesis we may assume that every class of dimension less than $k + 2n$ can be expressed uniquely in this way.

Let now $x \in H^{k+2n}(G_n)$. Again, by the main hypothesis, $\lambda(x) = p(c_1(E^{n-1}), ..., c_{n-1}(E^{n-1}))$ for some unique polynomial p. Thus, the element $x - p(c_1(E^n), ..., c_{n-1}(E^n))$ is in the kernel of λ, hence in the image of $z \mapsto z \smile c_n(E^n)$ by the short exact sequence. Thus,

$$x - p(c_1(E^n), ..., c_{n-1}(E^n)) = yc_n(E^n)$$

for some unique $y \in H^k(G_n)$. Since by the secondary induction hypothesis, y can be written uniquely as a polynomial $y = q(c_1(E^n), ..., c_n(E^n))$, we have

$$x = p(c_1(E^n), ..., c_{n-1}(E^n)) + q(c_1(E^n), ..., c_n(E^n))c_n(E^n).$$

If $x = p'(c_1(E^n), ..., c_{n-1}(E^n)) + q'(c_1(E^n), ..., c_n(E^n))c_n(E^n)$ for some p', q', then by applying λ, we have by the main induction hypothesis that $p' = p$, and since by the short exact sequence, $c_n(E^n)$ is not a zero divisor, we can divide the difference by $c_n(E^n)$ and deduce that $q' = q$. This shows that the polynomial expression of x is unique, and we have proved the theorem. □

5.3.4 Whitney Sum Formula

We will now prove a result concerning Chern classes analogous to the second part of Proposition 5.2.4.

Theorem 5.3.10. *Let B be a paracompact space, and let E_1 and E_2 be complex bundles over B, with ranks n and m respectively. Then*

$$c(E_1 \oplus E_2) = c(E_1)c(E_2).$$

Proof. The proof will be divided into two parts. First, we prove that there is a unique polynomial expression for $c(E_1 \oplus E_2)$ in terms of $c_1(E_1), ..., c_n(E_1), c_1(E_2), ..., c_m(E_2)$, which only depends on the ranks of the bundles, and after this we will show that this expression equals $c(E_1)c(E_2)$.

Consider first the case where the base space is $G_n \times G_m$, with projection maps

$$pr_1 : G_n \times G_m \to G_n \quad \text{and} \quad pr_2 : G_n \times G_m \to G_m.$$

Define two bundles over $G_n \times G_m$ by $E_1^n = pr_1^*(E^n)$ and $E_2^m = pr_2^*(E^m)$. By the Künneth formula, the cohomology ring $H^*(G_n \times G_m)$ is isomorphic to the tensor product $H^*(G_n) \otimes H^*(G_m)$, the isomorphism given by the cross product operation. Using Theorem 5.3.9, this tensor product in turn is isomorphic to $\mathbb{Z}[c_1(E^n), ..., c_n(E^n), c_1(E^m), ..., c_m(E^m)]$, the polynomial ring generated by the Chern classes of both bundles, with no polynomial relations among the generators. Thus, since the total Chern class $c(E^n \oplus E^m)$ is in this ring, there is a unique polynomial $p_{n,m}$ in $n + m$ variables such that

$$c(E^n \oplus E^m) = p_{n,m}(c_1(E^n), ..., c_n(E^n), c_1(E^m), ..., c_m(E^m)).$$

Let now B be any paracompact space, and let E_1 and E_2 be complex vector bundles over B of ranks n and m respectively. By Theorem 4.5.2, there exist maps $f : B \to G_n$ and $g : B \to G_m$ such that $f^*(E^n) = E_1$ and $g^*(E^m) = E_2$. Now define

$$h : B \to G_n \times G_m \quad \text{by} \quad h(b) = (f(b), g(b))$$

for all $b \in B$. Then $h^*(E_1^n) = E_1$ and $h^*(E_2^m) = E_2$, so that $h^*(E_1^n \oplus E_2^m) = E_1 \oplus E_2$, and h is clearly covered by a bundle map $E_1 \oplus E_2 \to E_1^n \oplus E_2^m$. By Proposition 5.3.4, we now have

$$\begin{aligned} c(E_1 \oplus E_2) &= h^*c(E_1^n \oplus E_2^m) \\ &= h^*p_{n,m}(c_1(E_1^n), ..., c_n(E_1^n), c_1(E_2^m), ..., c_m(E_2^m)) \\ &= p_{n,m}(h^*c_1(E_1^n), ..., h^*c_n(E_1^n), h^*c_1(E_2^m), ..., h^*c_m(E_2^m)) \\ &= p_{n,m}(c_1(E_1), ..., c_n(E_1), c_1(E_2), ..., c_m(E_2)). \end{aligned}$$

We must now calculate the polynomials $p_{n,m}$, or more precisely, show that

$$p_{n,m}(c_1, ..., c_n, c'_1, ..., c'_m) = (1 + c_1 + \cdots + c_n)(1 + c'_1 + \cdots + c'_m).$$

We will proceed by induction on $n + m$. If $n + m = 0$, then $n = m = 0$, and

$$1 = c(E_1^n \oplus E_2^m) = c(E_1^n)c(E_2^m) = 1 \cdot 1.$$

Thus, we can assume that

$$c(E_1^{n-1} \oplus E_2^m) = (1 + c_1(E_1^{n-1}) + \cdots + c_{n-1}(E_1^{n-1}))(1 + c_1(E_2^m) + \cdots + c_m(E_2^m)),$$

and similarly for $E_1^n \oplus E_2^{m-1}$. Let $T \to G_{n-1} \times G_m$ be a trivial line bundle. By Proposition 5.3.5, we have

$$\begin{aligned} c(E_1^{n-1} \oplus E_2^m) &= c(E_1^{n-1} \oplus T \oplus E_2^m) \\ &= p_{n,m}(c_1(E_1^{n-1} \oplus T), ..., c_n(E_1^{n-1} \oplus T), c_1(E_2^m), ..., c_m(E_2^m)) \\ &= p_{n,m}(c_1(E_1^{n-1}), ..., c_{n-1}(E_1^{n-1}), 0, c_1(E_2^m), ..., c_m(E_2^m)) \\ &= (1 + c_1(E_1^{n-1}) + \cdots + c_{n-1}(E_1^{n-1}))(1 + c_1(E_2^m) + \cdots + c_m(E_2^m)). \end{aligned}$$

This means that

$$p_{n,m}(c_1, ..., c_n, c'_1, ..., c'_m) = (1 + c_1 + \cdots + c_n)(1 + c'_1 + \cdots + c'_m) + u_1 c_n$$

for some unique polynomial u_1. Similarly, by changing the order of brackets in the expression for $c(E_1^{n-1} \oplus T \oplus E_2^m)$, we find that

$$p_{n,m}(c_1, ..., c_n, c'_1, ..., c'_m) = (1 + c_1 + \cdots + c_n)(1 + c'_1 + \cdots + c'_m) + u_2 c'_m$$

for some unique u_2. Since $\mathbb{Z}[c_1, ..., c_n, c'_1, ..., c'_m]$ is a unique factorization domain and c_n and c'_m are irreducible elements, these equations imply that

$$p_{n,m}(c_1, ..., c_n, c'_1, ..., c'_m) = (1 + c_1 + \cdots + c_n)(1 + c'_1 + \cdots + c'_m) + u c_n c'_m$$

for some unique u. By substituting $c(E_1^n \oplus E_2^m)$ into this equation, we see that u must have dimension zero, since otherwise the $n+m$-bundle $E_1^n \oplus E_2^m$ would have nonzero Chern classes in dimensions higher than $n + m$. Now, since the top Chern class equals the Euler class, using Proposition 5.2.4 we get

$$\begin{aligned} e(E_1^n \oplus E_2^m) &= c_{m+n}(E_1^n \oplus E_2^m) \\ &= (1 + u)c_n(E_1^n)c_m(E_2^m) = (1 + u)e(E_1^n)e(E_2^m) \\ &= (1 + u)e(E_1^n \oplus E_2^m). \end{aligned}$$

Since $e(E_1^n \oplus E_2^m)$ is nonzero and the cohomology ring $H^*(G_n \times G_m)$ is an integral domain, we have $u = 0$, so we have proved the theorem. □

As a corollary, we consider the case where E splits as a sum $E' \oplus T$, where T is a trivial bundle. This concludes our discussion on the relationship between triviality of a vector bundle an existence of linearly independent sections.

Corollary 5.3.11. *Let $\pi : E \to B$ be a vector bundle of rank n, and assume that it splits as the Whitney sum $E = E' \oplus T$, where T is a trivial bundle. Then $c(E) = c(E')$. In particular, if B is paracompact and Hausdorff and E possesses k linearly independent sections, then*

$$c_{n-k+1}(E) = c_{n-k+2}(E) = ... = c_n(E) = 0.$$

Proof. The first statement follows immediately from Proposition 5.3.5 and the above theorem. The second statement now follows from Remark 4.2.3, since in this case E indeed splits as a Whitney sum $E = E' \oplus T$, where T is a trivial bundle of rank k and E' has rank $n-k$, and so

$$\begin{aligned} c(E) &= 1 + c_1(E) + ... + c_{n-k}(E) + c_{n-k+1}(E) + ... + c_n(E) \\ &= c(E' \oplus T) = c(E')c(T) = c(E') \\ &= 1 + c_1(E') + ... + c_{n-k}(E'). \end{aligned}$$

Comparing dimensions in the expressions for $c(E)$ and $c(E')$ yields the result. □

In conclusion, Chern classes provide a powerful tool in the study of complex vector bundles, both due to their functorial properties as natural transformations, and the rich algebraic structure provided by the Whitney sum formula. The calculation of the cohomology ring of the Grassmannian G_n serves as a starting point for studying complex vector bundles over arbitrary base spaces, and is thus at the heart of the subject.

Although the subject is very classical and well understood, the author believes to have succeeded in clarifying and illuminating some technical arguments presented for example in [13]. In particular, the construction of the auxiliary bundle at the beginning of section 5.3.1 is merely mentioned in a passing remark in [13], and we have been able to provide the technical details of the construction.

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